Computer-Assisted Proofs in nonlinear dynamics based on spectral methods

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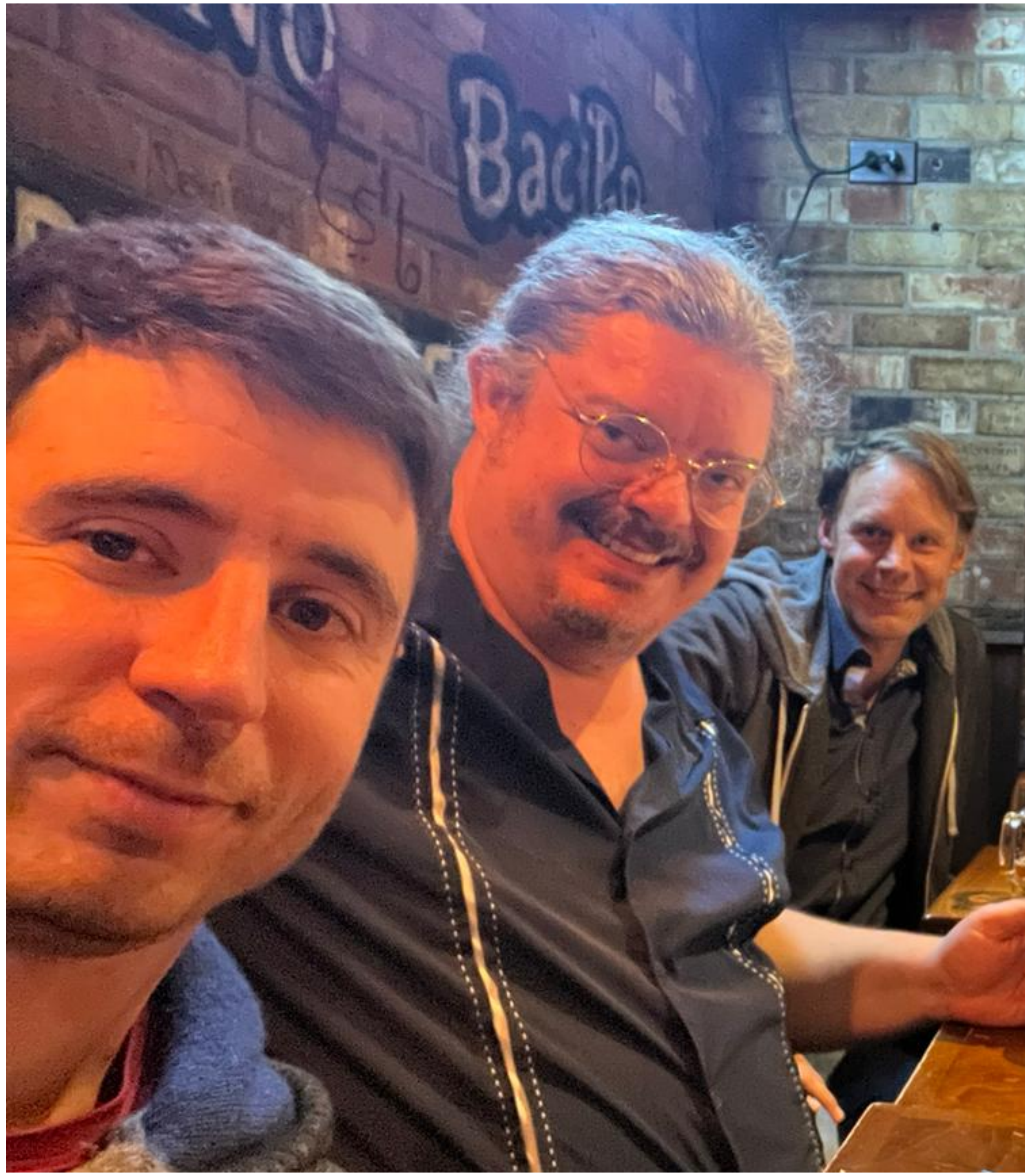
CAPs in dynamics: a spectral approach



joint work with

Jean-Philippe Lessard

Jay Mireles-James

Maxime Breden

CAP = Computer Assisted Proof

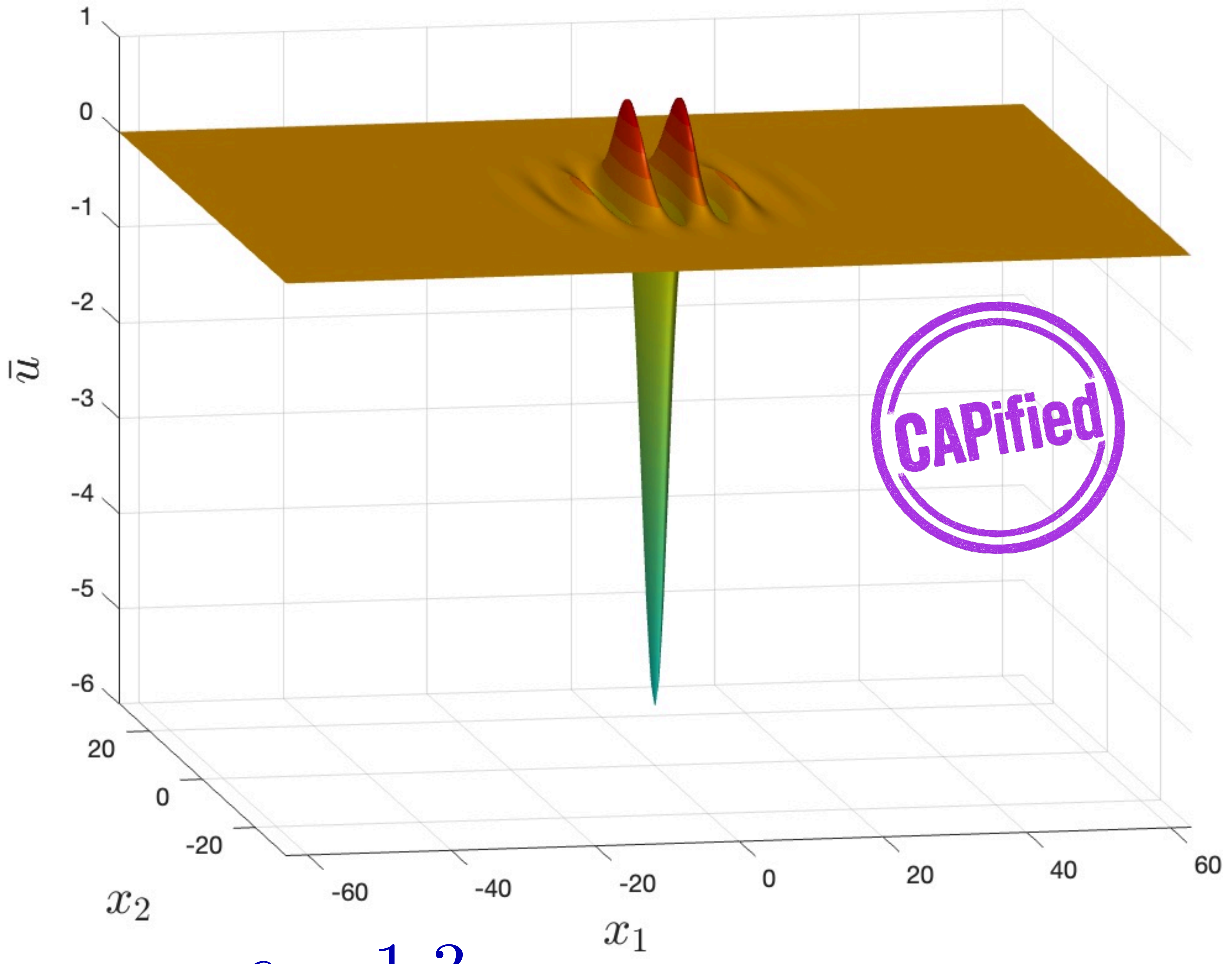
Example

Travelling wave in Suspension bridge equation

$$\frac{\partial^2 u}{\partial t^2} = -\Delta^2 u - e^u + 1$$

van der Aalst & Cadiot & Lessard & JB

Example



$c = 1.3$

$\bar{u}$ approximation with 60 x 20 Fourier modes

$\tilde{u}$ denotes solution

Theorem

$$\|\tilde{u} - \bar{u}\|_{C^0} \leq 0.0021$$

(6 minutes)

Finite periodic bridge: Fourier transform

$$\Delta^2 u + c^2 \partial_{x_1}^2 u + e^u - 1 = 0$$

$$u(x) = \sum_{n \in \mathbb{Z}^2} u_n e^{i\pi(n_1 x_1/d_1 + n_2 x_2/d_2)}$$

$$L_n u_n + G(u)_n = 0$$

$$G(u) = e^u - 1 - u$$

$$L_n = \pi^4 \left[\frac{n_1^2}{d_1^2} + \frac{n_2^2}{d_2^2}\right]^2 - c^2 \pi^2 \frac{n_1^2}{d_1^2} + 1$$

Use symmetry: restrict to cosine series

Newton-Kantorovich

general: $F(x) = 0$

$$F(u) := Lu + G(u) \qquad (Lu)_n = L_n u_n$$

$$F(u) := u + L^{-1}G(u)$$ Identity + Compact

CAP is based on Newton-Kantorovich: fixed point method

$$x \mapsto x - DF(x)^{-1}F(x)$$
$$x \mapsto x - DF(\overline{x})^{-1}F(x)$$
$$x \mapsto x - AF(x)$$

where A is an approximate inverse of $DF(\overline{x})$

with $\overline{x}$ an approximate zero of F

Newton-Kantorovich

Assume A is chosen so that it is injective

$T : x \mapsto x - AF(x)$

Assume you have derived computable bounds

$$\|AF(\bar{x})\| \leq Y$$

$$\|I - ADF(\bar{x})\| \leq Z_1$$

$$\|A[DF(x) - DF(\bar{x})]\| \leq Z_2\|x - \bar{x}\| \qquad \forall x \in B_{r_*}(\bar{x})$$

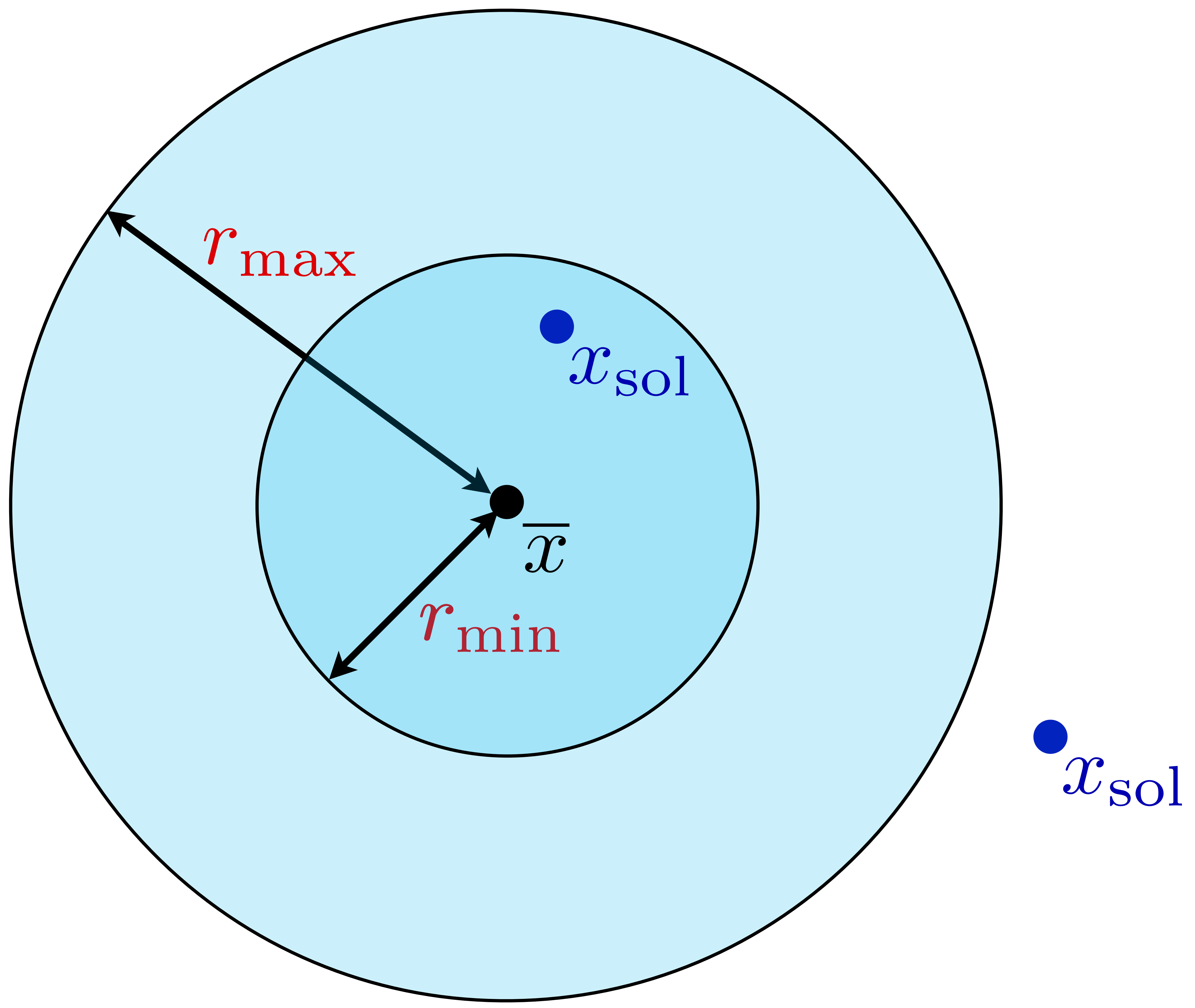
If for some $r_0 \leq r_*$

$$Y + Z_1 r_0 + \tfrac{1}{2} Z_2 r_0^2 \leq r_0$$ self-map

$$Z_1 + Z_2 r_0 < 1$$ contracts

Then $x \mapsto x - AF(x)$ is a contraction : F has unique zero in $B_{r_0}(\bar{x})$

Newton-Kantorovich



Newton-Kantorovich Theorem

Assume A is chosen so that it is injective

Assume you have derived computable bounds

$$\|AF(\bar{x})\| \le Y$$
$$\|I - ADF(\bar{x})\| \le Z_1$$
$$\|A[DF(x) - DF(\bar{x})]\| \le Z_2\|x - \bar{x}\| \qquad \forall x \in B_{r_*}(\bar{x})$$

If for some $r_0 \le r_*$ $\quad Y + Z_1 r_0 + \frac{1}{2} Z_2 r_0^2 \le r_0$

$$Z_1 + Z_2 r_0 < 1$$

Then $x \mapsto x - AF(x)$ is a contraction : F has unique zero in $B_{r_0}(\bar{x})$

Banach spaces of real analytic periodic functions

$$u(\theta) = \sum_{\substack{n \in \mathbb{Z} \\ n \in \mathbb{Z}^d}} u_n e^{in\theta}$$

$$\|u\|_\nu = \sum_{n \in \mathbb{Z}} |u_n| \, \nu^{|n|} \qquad \nu > 1$$

Banach space ℓ^1_ν

Natural product $(a * b)_k = \sum_{n \in \mathbb{Z}} a_n b_{k-n}$

Banach algebra $\|a * b\|_\nu \leq \|a\|_\nu \|b\|_\nu$

Numerics

Projection $(\pi^{\leq N} u)_n = \begin{cases} u_n & |n| \leq N \\ 0 & n > |N| \end{cases}$ $\qquad \pi^{>N} := I - \pi^{\leq N}$

Truncated zero finding problem $F^{\leq N} : \pi^{\leq N} \ell^1_\nu \to \pi^{\leq N} \ell^1_\nu$

With approximate zero $\bar{u} \in \pi^{\leq N} \ell^1_\nu$

Find numerical inverse $A^{\leq N}$ of the Jacobian $DF^{\leq N}(\bar{u})$

Approximate inverse of $DF(\bar{u})$ is $A = A^{\leq N} \pi^{\leq N} + I^{>N} \pi^{>N}$

CAP: Residue bound

$\|AF(\bar{x})\| \leq Y$

where $F(\bar{x}) = \bar{x} + L^{-1}G(\bar{x}) = \bar{x} + K(\bar{x})$

- compute approximation $\overline{K}$ of $K(\bar{x})$ with $\|\overline{K} - K(\bar{x})\| \leq \delta_K$

$$\begin{aligned}\|AF(\bar{x})\| = \|A[\bar{x} + K(\bar{x})]\| &\leq \|A[\bar{x} + \overline{K}]\| + \|A[K(\bar{x}) - \overline{K}]\| \\ &\leq \underbrace{\|A[\bar{x} + \overline{K}]\| + \|A\|\delta_K}_{Y}\end{aligned}$$

CAP: Precondition bound

$$\|I - ADF(\bar{x})\| \leq Z_1$$

- compute approximation $\overline{DK}$ of $DK(\bar{x})$ with $\|\overline{DK} - DK(\bar{x})\| \leq \delta_{DK}$
- and $\|\pi^{>N}\overline{DK}\| \leq C(N) \to 0$ as $N \to \infty$
- where $\overline{DK}$ is a finite bandwidth operator

$$\begin{aligned}
\|I - ADF(\bar{x})\| &\leq \|I - A[I + \overline{DK}]\| + \|A[DK(\bar{x}) - \overline{DK}]\| \\
&\leq \|(\pi^{\leq N} + \pi^{>N})(I - A[I + \overline{DK}])\| + \|A\|\delta_{DK} \\
&\leq \|\pi^{\leq N} - A^{\leq N}\pi^{\leq N}[I + \overline{DK}]\| + \|\pi^{>N}\overline{DK}\| + \|A\|\delta_{DK}
\end{aligned}$$

Triangle inequality

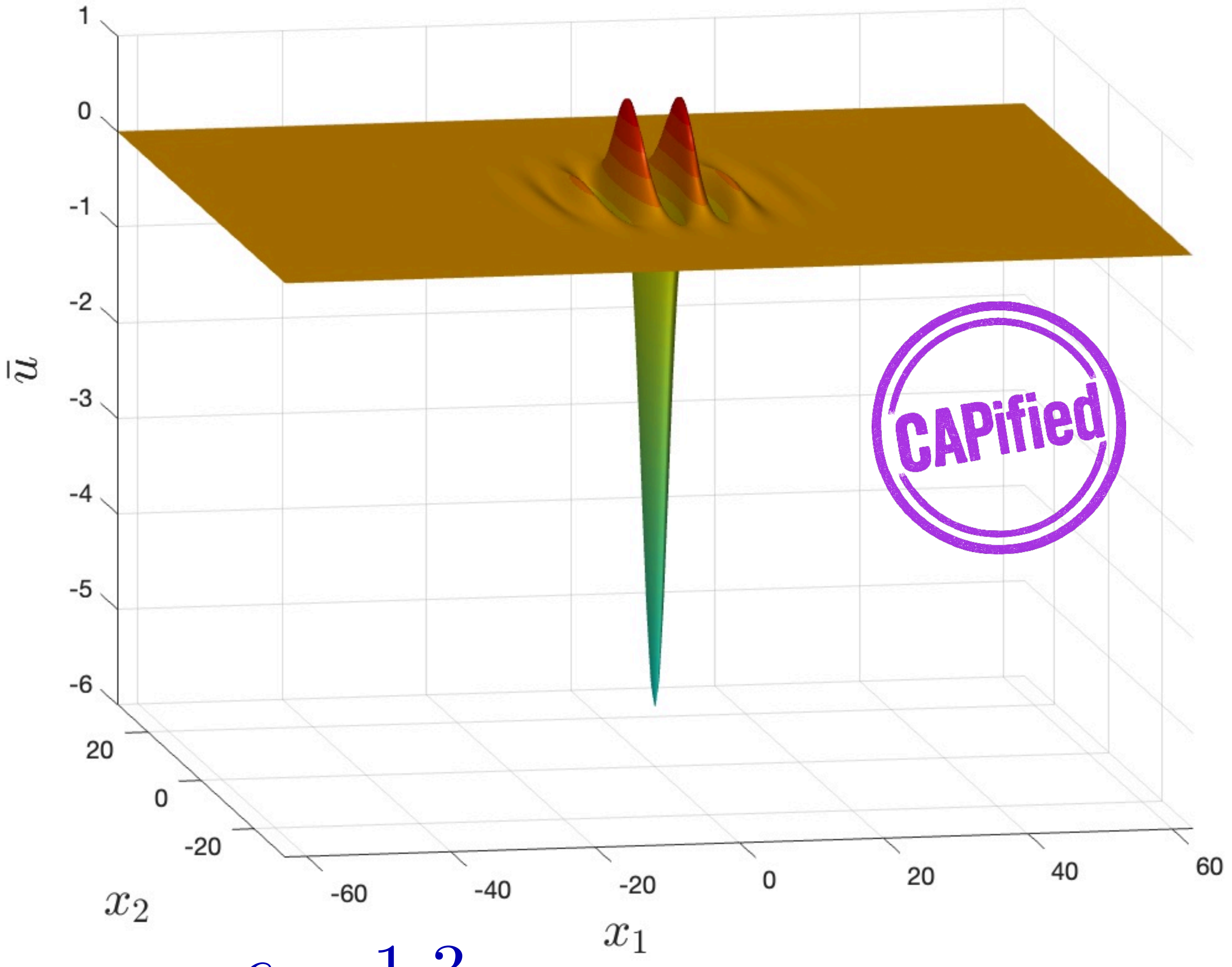
CAP: Curvature bound

$$\|A[DF(x) - DF(\bar{x})]\| \leq Z_2 \|x - \bar{x}\| \qquad \forall x \in B_{r_*}(\bar{x})$$

$$\begin{aligned} \|A[DF(x) - DF(\bar{x})]\| &= \|A[DK(x) - DK(\bar{x})]\| \\ &= \|AL^{-1}[DG(x) - DG(\bar{x})]\| \\ &\leq \|AL^{-1}\| \|DG(x) - DG(\bar{x})\| \end{aligned}$$

$$\begin{aligned} \|DG(x) - DG(\bar{x})\| &= \|DG(\bar{x} + (x - \bar{x})) - DG(\bar{x})\| \\ &\leq \sup_{s \in B_{r_*}} \|D^2 G(\bar{x} + s)\| \, \|x - \bar{x}\| \end{aligned}$$

Example



$c = 1.3$

$\bar{u}$ approximation with 60 x 20 modes

- $Y = 0.00055$
- $Z_1 = 0.66$
- $Z_2 = 74$

$$r_0 = \frac{(1-Z_1)-\sqrt{(1-Z_1)^2-2YZ_2}}{Z_2} = 0.0021$$

Theorem:

there is a solution $\tilde{u}$ with $\|\tilde{u} - \bar{u}\|_{C^0} \leq r_0$

$\bar{u}$ approximation with 100 x 60 modes: $r_0 = 0.0000022$

Some more CAPs in PDEs

Navier Stokes equations

$$
\begin{cases}
\dfrac{\partial \vec{\mathbf{u}}}{\partial t} + (\vec{\mathbf{u}} \cdot \nabla)\vec{\mathbf{u}} - \nu \nabla^2 \vec{\mathbf{u}} + \dfrac{1}{\rho}\nabla p = \vec{f} \\
\nabla \cdot \vec{\mathbf{u}} = 0
\end{cases}
$$

CAPified

CAP work on fluid flows:

Nakao, Plum, Watanabe

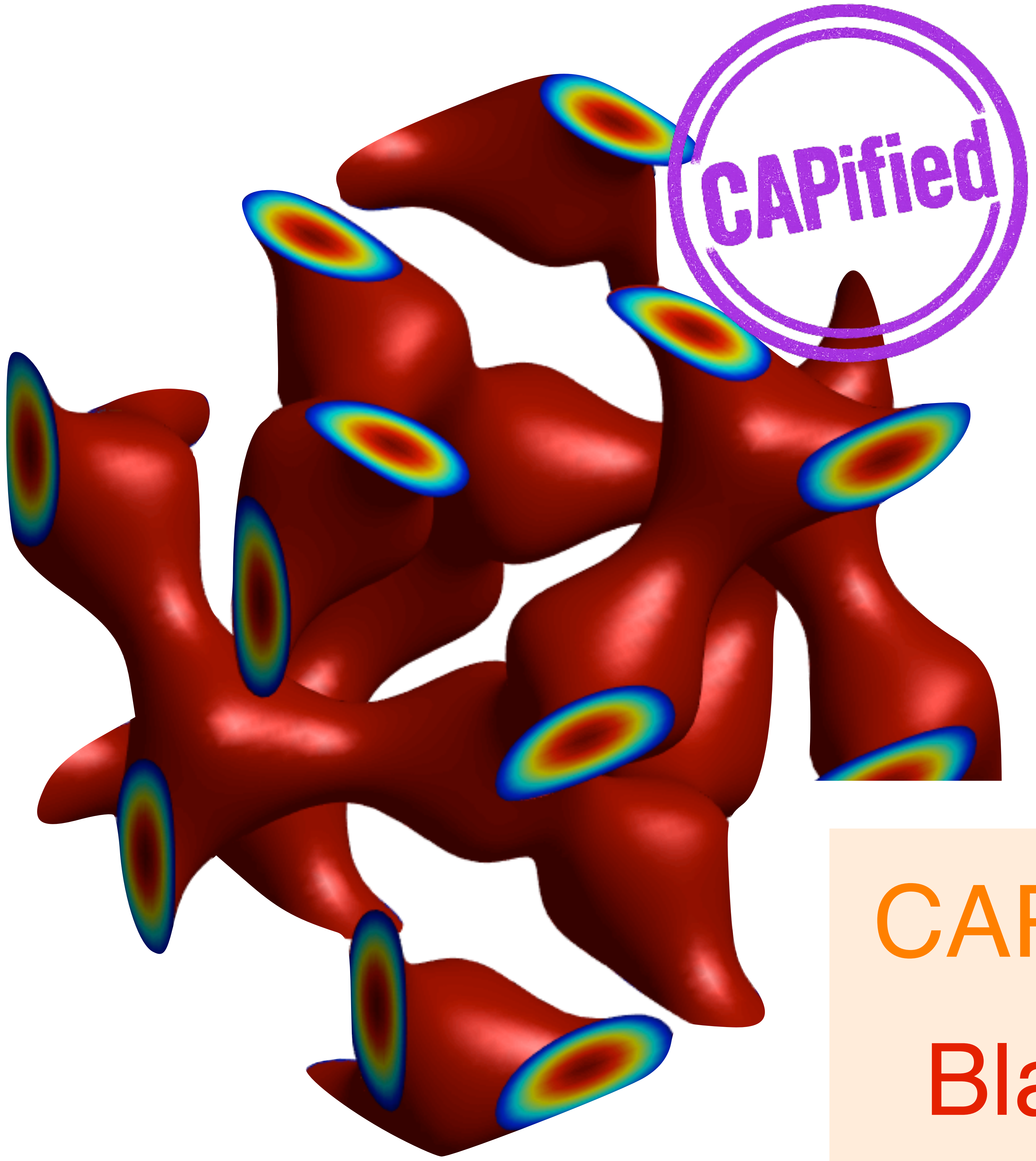
Gomez-Serrano

Breden & Lessard & van Veen & JB

PDEs with symmetry

Ohta-Kawasaki equation

$$-\Delta(\gamma^{-2}\Delta u + u - u^3) - \sigma(u - m) = 0$$



CAP work on symmetries:

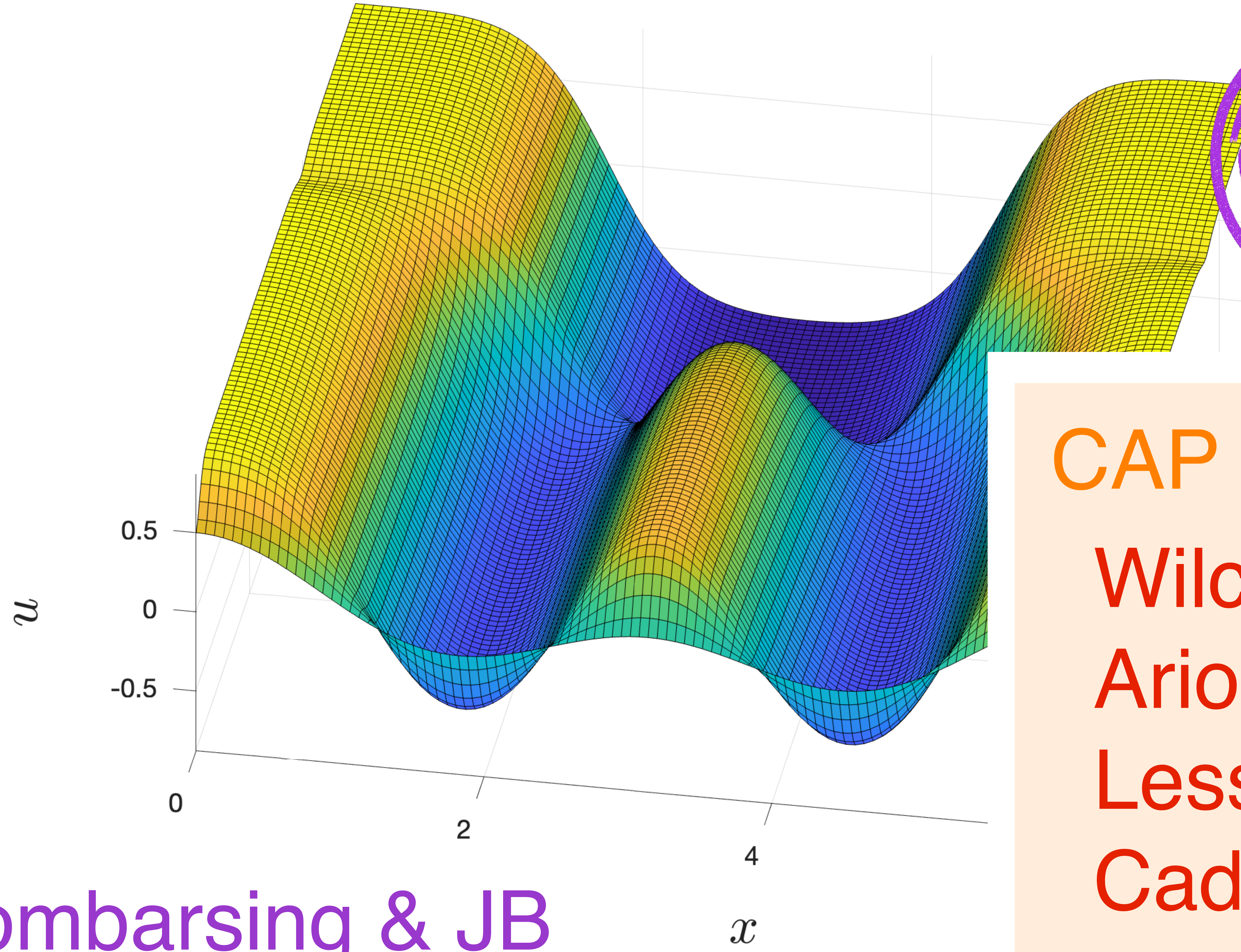
Blanco-Cadiot

Williams & JB

Initial Value Problem for PDE

Ohta-Kawasaki equation

$$\frac{\partial u}{\partial t} = -\Delta(\gamma^{-2}\Delta u + u - u^3) - \sigma(u - m)$$



CAP work on IVPs:

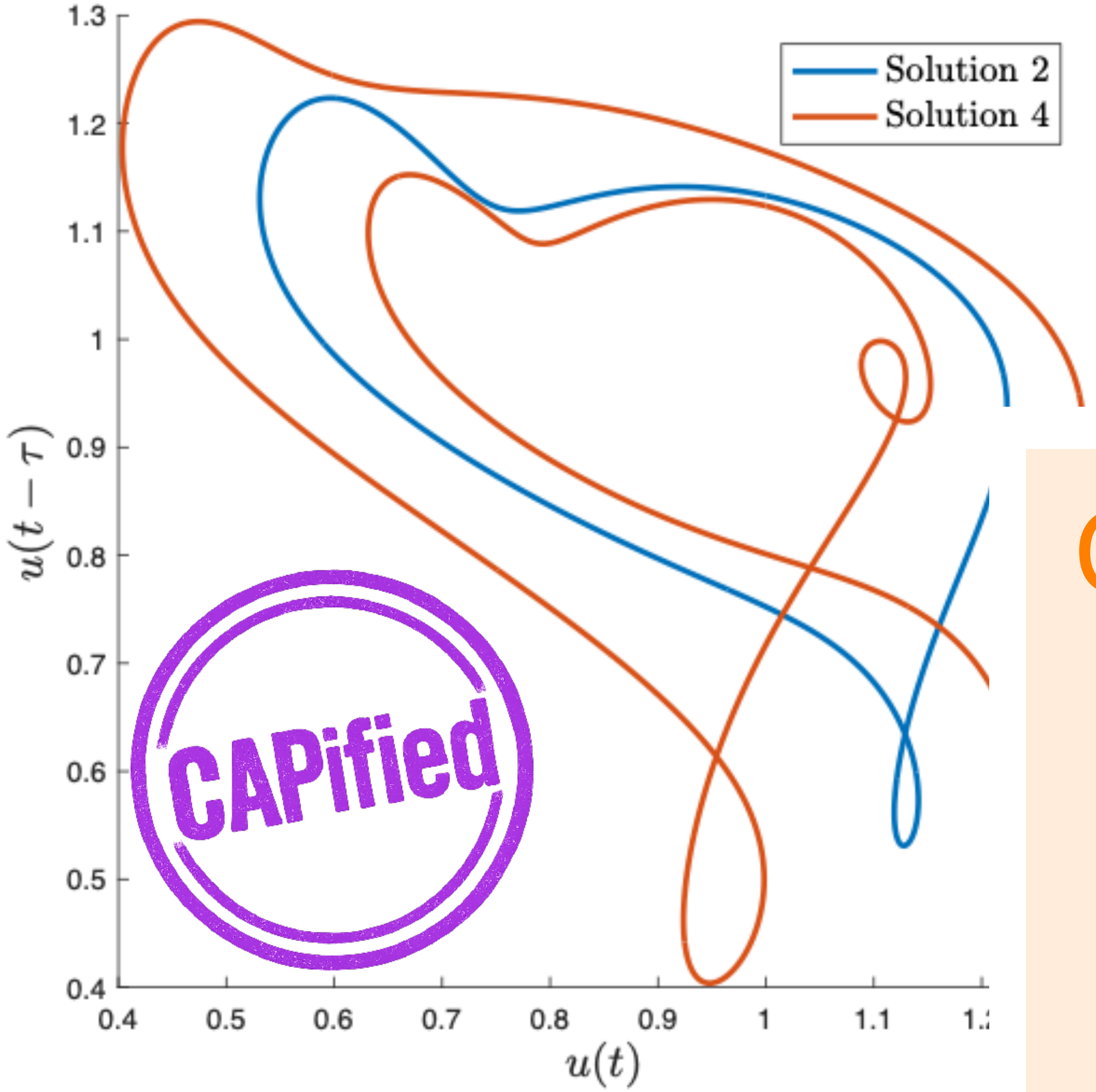
- Wilczak-Zgliczyński
- Arioli-Koch
- Lessard-Takayasu
- Cadiot-Cyranka-Lessard

Breden & Sheombarsing & JB

Delay Differential Equations

Mackey-Glass equation

$$u'(t) = -u(t) + 2\frac{u(t-\tau)}{1+u(t-\tau)^{\rho}}$$



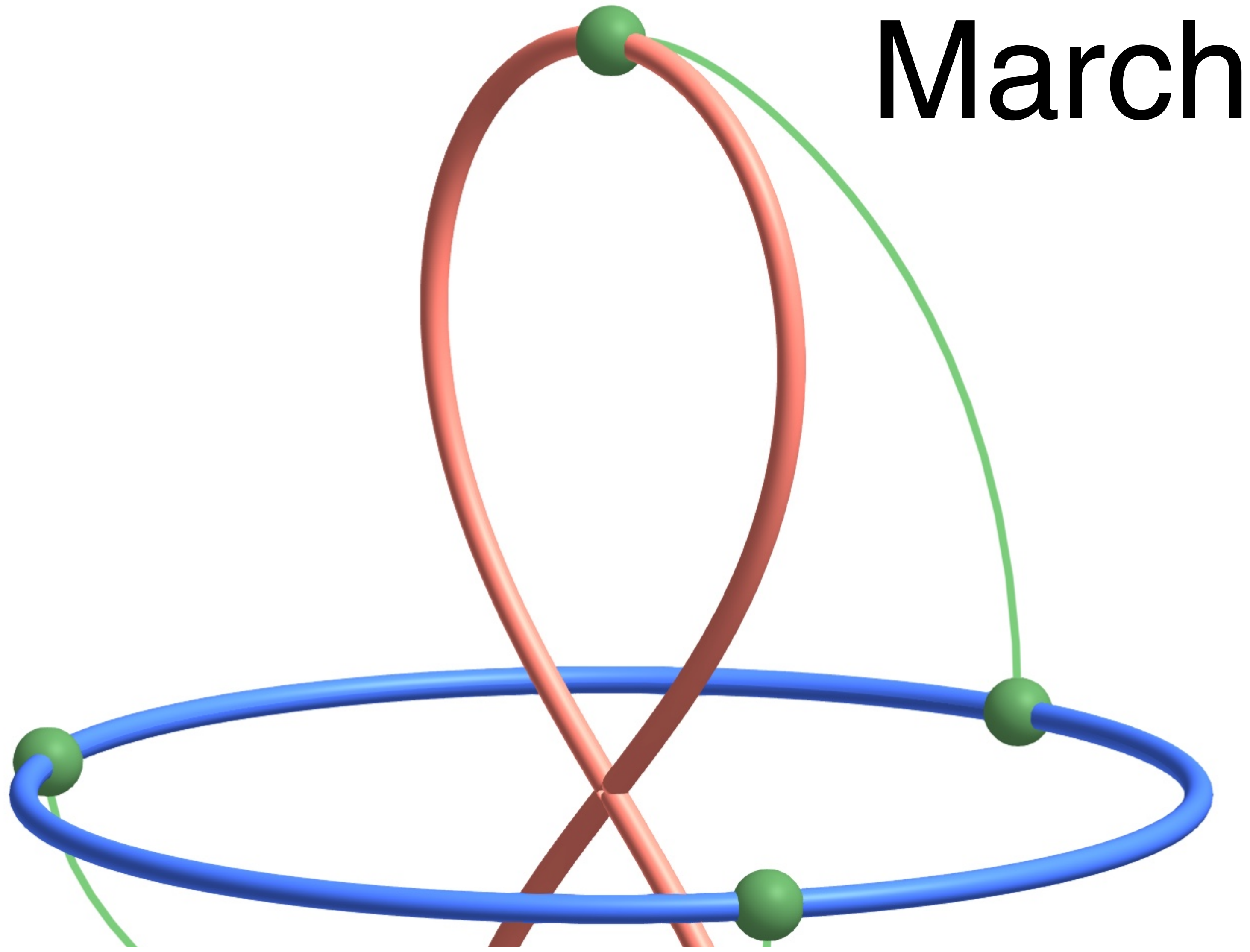
CAP work on DDEs:

- Jaquette
- Szczlina-Zgliczyński
- Hénot-Lessard-Mireles
- Breden-Cadiot-Church-…

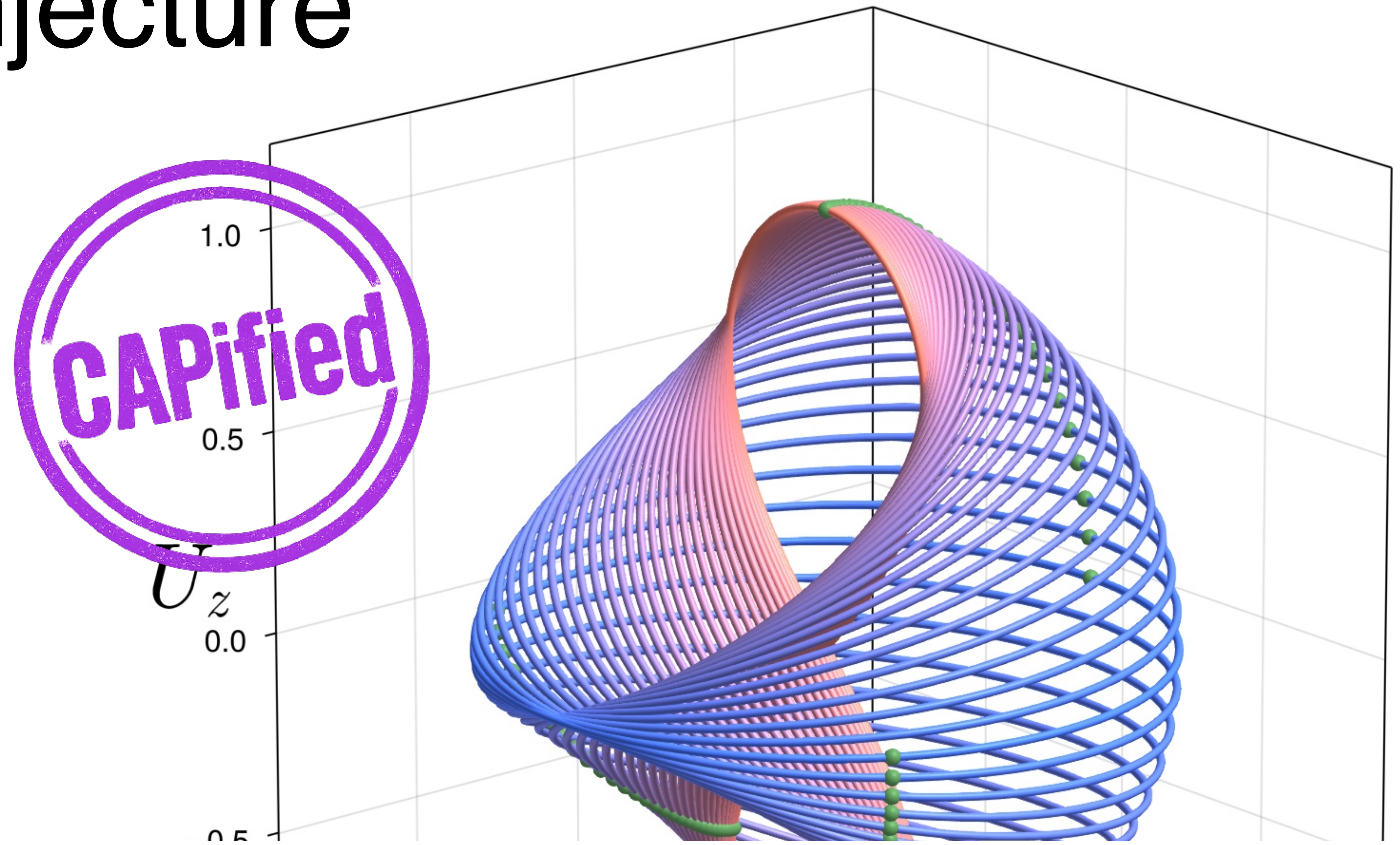
Groothedde & Lessard & JB

Celestial Mechanics

Three body problem (choreographies)



Marchal's conjecture



Integrators and more:

CAPD: Kapela-Wilczak-Zgliczyński-…

CAP work on KAM tori:

Figueras-Haro

Calleja & García-Azpeitia & Hénot & Lessard & Mireles James

Simple spectral example

IVP $\begin{cases} u' = G(u) \\ u(0) = 0 \end{cases}$

Taylor $u(t) = \sum_{n \in \mathbb{N}} a_n t^n$

$n a_n = G(a)_{n-1}$

$$\boxed{a - \Lambda G(a) = 0}$$

$(\Lambda b)_n = \begin{cases} 0 & n = 0 \\ \frac{1}{n} b_{n-1} & n > 0 \end{cases}$ Integration

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zeros of $F(a) = a + K(a)$ Identity + Compact

The main question is how to compute $G(a)$ in general

Another simple spectral example

BVP $\begin{cases} u'' = G(u) = \dfrac{\lambda}{(1+u)^2} \\ u(-1) = 0 = u(1) \end{cases}$ MEMS model

Chebyshev $u(s) = a_0 T_0(s) + 2\sum_{n=1}^{\infty} a_n T_n(s)$

$$\boxed{a - \Lambda G(a) = 0}$$

$$(\Lambda b)_n = \begin{cases} \text{“stuff”} & n = 0, 1 \\ \frac{1}{4}\left(\frac{1}{n^2-n} b_{n-2} - \frac{2}{n^2-1} b_n + \frac{1}{n^2+n} b_{n+2}\right) & n > 1 \end{cases}$$

The “stuff” hides that not everything is simple algebra

Chebyshev is Fourier in disguise: $T_n(s) = \cos(n \arccos(s))$

Taylor is half of Fourier up to rescaling

Error propagation for Fourier

$$G\Big(\sum_{n\in\mathbb{Z}} u_n e^{in\theta}\Big) = \sum_{n\in\mathbb{Z}} G(u)_n e^{in\theta}$$

any locally analytic nonlinearity

$$\begin{tikzcd}
u \arrow[r] \arrow[d] & \bar{u} \arrow[d] \arrow[dr] & \\
G(u) & G(\bar{u}) \arrow[l] & \overline{G(\bar{u})} \arrow[l]
\end{tikzcd}$$

$\bar{u} \in \pi^{\leq N}\ell^1_\nu$ approximates $u \in \ell^1_\nu$

with error bound $\|u - \bar{u}\| \leq \delta_u$

$$\overline{G(\bar{u})} = \texttt{fft} \circ G \circ \texttt{ifft}(\bar{u})$$ of size $N_{\texttt{fft}}$

$$\|G(u) - \overline{G(\bar{u})}\| \leq \|G(u) - G(\bar{u})\| + \|G(\bar{u}) - \overline{G(\bar{u})}\|$$

Example that is (too) tractable

$G(u) = u^2$

$$G(u)_n = \sum_{k \in \mathbb{Z}} u_k u_{n-k} = (u * u)_n$$

Banach algebra $\|a * b\|_\nu \leq \|a\|_\nu \|b\|_\nu$

if $\bar{u} \in \pi^{\leq N} \ell^1_\nu$ then $\bar{u} * \bar{u} \in \pi^{\leq 2N} \ell^1_\nu$

Choose $N_{\texttt{fft}} \geq 2N$ then $\|G(\bar{u}) - \overline{G(\bar{u})}\| =$ floating point rounding errors controlled by interval arithmetic

$$\begin{aligned}
\|G(u) - G(\bar{u})\| &= \|u * u - \bar{u} * \bar{u}\| \\
&= \|(u + \bar{u}) * (u - \bar{u})\| \\
&\leq \|(u + \bar{u})\| \|(u - \bar{u})\| \\
&\leq \|2\bar{u} + (u - \bar{u})\| \delta_u \\
&\leq (2\|\bar{u}\| + \delta_u) \delta_u
\end{aligned}$$

Error propagation for Fourier: aliasing

- Combine contour integral & Poisson summation formula to estimate the aliasing error between $G(\bar{u})$ and $\overline{G(\bar{u})}$

$$\left|\left[G(\bar{u}) - \overline{G(\bar{u})}\right]_n\right| \leq C_{\bar{\nu}} \frac{\bar{\nu}^n + \bar{\nu}^{-n}}{\bar{\nu}^{N_{\mathtt{fft}}} - 1}$$

for n not too large

$$C_{\bar{\nu}} = \frac{1}{2\pi} \int_0^{2\pi} |G(\bar{u}(\theta \pm i \log \bar{\nu}))| d\theta$$

Explicit control on $\|G(\bar{u}) - \overline{G(\bar{u})}\|_\nu$

- Already in a 2017 paper by Figueras & Haro & Luque
- building on early work by de la Llave & Rana
- CAP for KAM tori results using Nash-Moser setup
- always giving up a bit of regularity

Error propagation for Fourier: the other term

How about $\|G(u) - G(\bar{u})\|_\nu$?

- Is $G(u)$ even in ℓ^1_ν for $u \in \ell^1_\nu$?
- Can we estimate $\|G(u)\|_\nu$ in terms of $\|u\|_\nu$?
- Obviously need to avoid poles of G

This question is too general. We only need this for u close to $\bar{u}$

Still:
- we do not want to give up any bit of regularity
- and we need explicit error control

Let's focus on what we need for the CAP in the curvature bound:

$$\sup_{\|h\|_\nu \leq r_*} \|G''(\bar{u} + h)\|_\nu$$

Error propagation for Fourier: the other term

$$\sup_{\|h\|_\nu \le r_*} \|G''(\bar{u}+h)\|_\nu \quad \text{call } \widehat{G} = G''$$

$$\widehat{G}(\bar{u}+h) = \sum_{k=0}^{\infty} \frac{\widehat{G}^{(k)}(\bar{u})}{k!} h^k$$

with h^k in the convolution sense. Banach algebra property gives:

$$\|\widehat{G}(\bar{u}+h)\|_\nu \le \sum_{k=0}^{\infty} \frac{\|\widehat{G}^{(k)}(\bar{u})\|_\nu}{k!} \|h\|_\nu^k \le \sum_{k=0}^{\infty} \frac{\|\widehat{G}^{(k)}(\bar{u})\|_\nu}{k!} r_*^k$$

This is good since it uses exactly the information on h that we have

But is requires control on all derivatives of $\widehat{G}$

Complex analysis lemma

$$\widehat{G}(\bar{u}+h) = \widehat{G}(\bar{u}(z)+h(z)) = \sum_{k=0}^{\infty} \frac{\widehat{G}^{(k)}(\bar{u}(z))}{k!} h(z)^k$$

For each fixed z

$$\widehat{G}^{(k)}(\bar{u}(z)) = \frac{k!}{2\pi i} \oint_{B_{r_\circ}(\bar{u}(z))} \frac{\widehat{G}(w)}{w^{k+1}} dw$$

$$= \frac{k!}{2\pi r_\circ^k} \int_0^{2\pi} \widehat{G}(\bar{u}(z) + r_\circ e^{i\phi}) e^{-ik\phi} d\phi$$

Arbitrary z ! Triangle inequality for Bochner integrals:

$$\|\widehat{G}^{(k)}(\bar{u})\|_\nu \leq \frac{k!}{2\pi r_\circ^k} \int_0^{2\pi} \|\widehat{G}(\bar{u} + r_\circ e^{i\phi})\|_\nu d\phi$$

Complex analysis lemma

$$\|\widehat{G}^{(k)}(\bar{u})\|_\nu \leq \frac{k!}{2\pi r_\circ^k}\int_0^{2\pi} \|\widehat{G}(\bar{u}+r_\circ e^{i\phi})\|_\nu d\phi$$

Let $\widehat{C}$ bound $\max_{\phi\in[0,2\pi]}\|\widehat{G}(\bar{u}+r_\circ e^{i\phi})\|_\nu$

$$\sup_{\|h\|_\nu\leq r_*} \|\widehat{G}(\bar{u}+h)\|_\nu \leq \sum_{k=0}^{\infty}\frac{\|\widehat{G}^{(k)}(\bar{u})\|_\nu}{k!} r_*^k$$

$$\leq \sum_{k=0}^{\infty}\frac{k!\widehat{C}}{r_\circ^k}\frac{r_*^k}{k!}$$

$$= \widehat{C}\frac{1}{1-\frac{r_*}{r_\circ}}$$

for $r_* < r_\circ$

Avoiding singularities of G

- use winding numbers (avoiding poles)
  and open mapping theorem (avoiding branch cuts)
- a lot, but not everything, can be automated:
  you need to know where the singularities of G are (not)

For systems, PDEs and continuation:

- nonlinearities depending on on multiple variables $G(u_1(z), u_2(z))$
- unknown functions depending on multiple variables $G(u(z_1, z_2))$

Conclusions

CAPified

- CAPs can be used to turn simulations into proofs
- Based on quantitative fixed point theorem (Newton-Kantorovich)
- Locally analytic: general Fourier-Taylor-Chebyshev spectral method
- Bounds: interpolation estimates and Fourier/complex analysis
- Error control in computations through interval arithmetic
- Reuse code (Matlab, Julia [Hénot]): less time spent on "technique"
- Concentrate on what is the specific math for the problem
  (it is still nonlinear analysis: each problem has its own tweak)
- And plenty of problems do not fall into the category discussed:
  lots of CAPs with other bases, spaces, unbounded domains, etc.

Thank you!

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