

Existence of cycles of renormalisation operators for coupled systems

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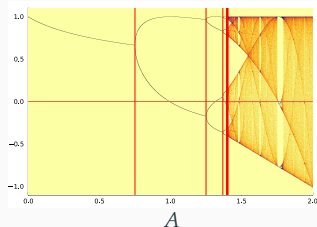
Introduction

Universality in Period-doubling Cascades

One-parameter families of unimodal maps, e.g.

$$f : x \mapsto 1 - Ax^2, \quad x \in [-1, 1], \quad A \in [0, 2].$$

Quadratic critical point (local maximum) at the origin.



Universal scaling:

$$\delta := \lim_{n \rightarrow \infty} \frac{A_{n+1} - A_n}{A_{n+2} - A_{n+1}} = 4.669 \dots$$

$$\alpha := \lim_{n \rightarrow \infty} \frac{\varepsilon_n}{\varepsilon_{n+1}} = -2.502 \dots$$

Cvitanović-Feigenbaum operator

$$R(g)(x) := a^{-1}g(g(ax)), \quad \text{where } a := g(g(0)).$$

[Feigenbaum. Journal of Statistical Physics, 19(1):25–52, 1978.]

Universal scaling

'Feigenbaum' scaling:

- Double the period,
- Scale A by $\frac{1}{\delta}$ with $\delta \approx 4.669$,
- Scale x by $\frac{1}{\alpha}$ with $\alpha \approx -2.502$.

'C-type' scaling:

- Quadruple the period,
- Scale A, C by $\frac{1}{\delta_1}$ and $\frac{1}{\delta_2}$ with $\delta_1 \approx 92.4$ and $\delta_2 \approx 4.19$,
- Scale x, y by $\frac{1}{\alpha^*}$ and $\frac{1}{\beta^*}$ with $\alpha^* \approx 6.57$ and $\beta^* \approx 22.1$.

Coupled systems

Coupled systems: doubling operator

Cycles of the doubling operator acting on **pairs of maps of two variables**:

$$R(g,f)(x,y) = \begin{pmatrix} a^{-1}g(g(ax, by), f(ax, by)) \\ b^{-1}f(g(ax, by), f(ax, by)) \end{pmatrix},$$

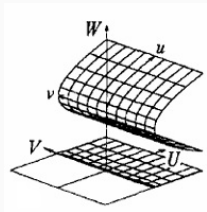
where $g, f : \mathbb{R}^2 \rightarrow \mathbb{R}$ are smooth, with

$$\begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} g(g(0,0), f(0,0)) \\ f(g(0,0), f(0,0)) \end{pmatrix}$$

which enforces the normalisation $R(g,f)(0,0) = (1,1)$.

C-type systems and universality

Critical points in non-invertible maps $\mathbb{R}^2 \rightarrow \mathbb{R}^2$; arising from fold type critical point: $(x, y) \mapsto (x^2, y)$:



Example prototypical three-parameter family

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} g(x, y) \\ f(x, y) \end{pmatrix} := \begin{pmatrix} A - x^2 + Cy \\ -x^2 + By \end{pmatrix}.$$

Cycles $\{(x_k, y_k)\}$ of period 4^n and $2 \cdot 4^n$ (unstable) and vice versa.

Ansatz for C-type system (in ‘scaling coordinates’)

Ansatz (relevant to “C-type” universality): $(X, Y) := (x^2, y)$,

$$g(x, y) = G(X, Y), \quad f(x, y) = F(X, Y).$$

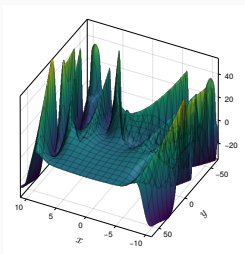
The equivalent operator T :

$$\begin{aligned} R(g, f)(x, y) &= T(G, F)(X, Y) \\ &:= \begin{pmatrix} a^{-1}G(G(a^2X, bY)^2, F(a^2X, bY)) \\ b^{-1}F(G(a^2X, bY)^2, F(a^2X, bY)) \end{pmatrix}, \end{aligned}$$

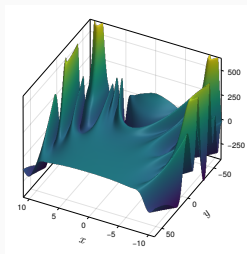
$$\text{with } \begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} G(G(0,0)^2, F(0,0)) \\ F(G(0,0)^2, F(0,0)) \end{pmatrix}.$$

Universality of C-type behaviour is explained by the **existence of a 2-cycle** $(g_1, f_1), (g_2, f_2)$ **of** R (hence T).

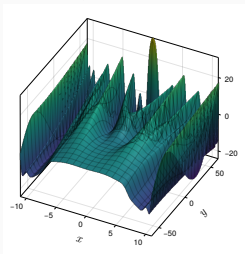
Good approximate 2-cycle functions (g_1, f_1) and (g_2, f_2)



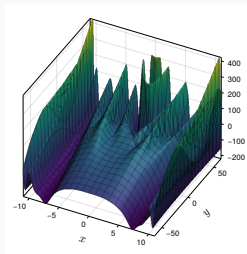
$g_1(x, y)$



$f_1(x, y)$



$g_2(x, y)$



$f_2(x, y)$

Rigorous computations in the function space

A suitable function space: the polydisc algebra

Let $G_m, F_m \in \mathcal{A} = \mathcal{A}(\Omega_m)$, the space of functions analytic on a bidisc

$$\Omega_m := D(c_m, r_m) \times D(d_m, s_m) \subset \mathbb{C}^2,$$

and continuous on $\overline{\Omega_m}$, with finite ℓ_1 -norm (of coefficients).

Such maps have a convergent **power series** on Ω_m :

$$G(X, Y) = \sum_{j,k=0}^{\infty} G_{jk} e_{jk}(X, Y) = \sum_{j,k=0}^{\infty} G_{jk} \left(\frac{X - c_m}{r_m} \right)^j \left(\frac{Y - d_m}{s_m} \right)^k.$$

Formation of a Banach space

Vector space Operations:

$$(aG + bF)(X, Y) := aG(X, Y) + bF(X, Y),$$

$$(G \cdot F)(X, Y) := G(X, Y) F(X, Y),$$

$$H \circ (G \oplus F)(X, Y) := H(G(X, Y), F(X, Y)).$$

Banach algebra with Schauder basis (e_{jk}) and ℓ_1 -norm

$$\|H\|_{\Omega_m} = \|H\|_1 := \sum_{j,k=0}^{\infty} |H_{jk}|.$$

Domain extension for T

We use two domains:

$$P_1 \in \Omega_1 = D(c_1, r_1) \times D(d_1, s_1) \text{ and } P_2 \in \Omega_2 = D(c_2, r_2) \times D(d_2, s_2).$$

For the 2-cycle, we can write $T_1(P_1) = P_2$ and $T_2(P_2) = P_1$:

$$\begin{aligned} T_1(G_1, F_1)(X, Y) &= (a_1^{-1}, b_1^{-1})(\mathbf{G}_1, \mathbf{F}_1) \overbrace{(\mathbf{G}_1^2, \mathbf{F}_1)(a_1^2, b_1)}^{\Theta_1(X, Y)}(X, Y) \\ &= (G_2, F_2)(X, Y) \text{ on } \Omega_2 \end{aligned}$$

$$\begin{aligned} T_2(G_2, F_2)(X, Y) &= (a_2^{-1}, b_2^{-1})(\mathbf{G}_2, \mathbf{F}_2) \overbrace{(\mathbf{G}_2^2, \mathbf{F}_2)(a_2^2, b_2)}^{\Theta_2(X, Y)}(X, Y) \\ &= (G_1, F_1)(X, Y) \text{ on } \Omega_1. \end{aligned}$$

We need to be sure that T is **well-defined and differentiable** on our domains.

For $(G_1, F_1) \in \mathcal{A}(\Omega_1)^2$ and $(G_2, F_2) \in \mathcal{A}(\Omega_2)^2$, we must have

$$\begin{aligned}\overline{(a_1^2, b_1)\Omega_2} &\subset \Omega_1, \\ \overline{(G_1^2, F_1) \left((a_1^2, b_1)\Omega_2 \right)} &\subset \Omega_1, \\ \overline{(a_2^2, b_2)\Omega_1} &\subset \Omega_2, \\ \overline{(G_2^2, F_2) \left((a_2^2, b_2)\Omega_1 \right)} &\subset \Omega_2.\end{aligned}$$

By insisting that the closures of the left-hand sides are contained in the relevant domains, we ensure that the maps have analytic extensions to strictly larger domains and that the derivatives $DT_1(P_1)$ and $DT_2(P_2)$ are compact.

Centres and radii improvements

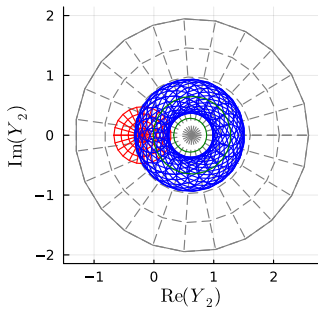
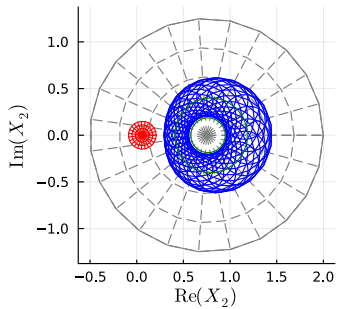
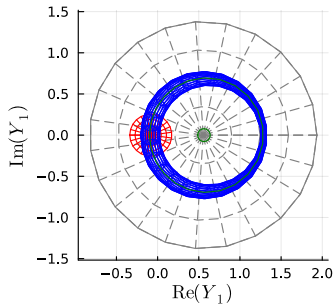
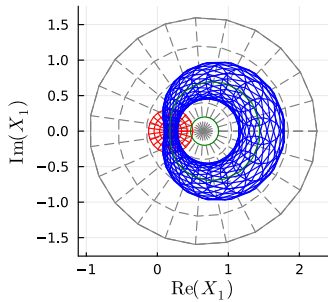
Centres: For a 2-cycle $((G_1, F_1), (G_2, F_2))$, if centre pairs (c_1, d_1) and (c_2, d_2) are exchanged by the 'inner' maps Θ_1 and Θ_2 , then (c_1, d_1) and (c_2, d_2) are roots of (G_1, F_1) and (G_2, F_2) respectively.

Radii: We want to minimise certain norms, for example

$$\max \left(\frac{1}{r_1} |a_1^2 c_2 - c_1| + \frac{r_2}{r_1} a_1^2, \frac{1}{s_1} |b_1 d_2 - d_1| + \frac{s_2}{s_1} |b_1| \right) < 1.$$

Results: We got the following values after optimising the centres and experimenting with different radii:

$$\begin{aligned} (c_1, d_1) &\simeq (0.64623, 0.55421), & (r_1, s_1) &= (1.6, 1.38), \\ (c_2, d_2) &\simeq (0.74835, 0.63468), & (r_2, s_2) &= (1.25, 1.95). \end{aligned}$$



Rigorous computations

Using intervals coefficients, define the **convex closed function ball** $\mathcal{B}(v_P, v_H, v_E) \subset \mathcal{A}$ by

$$\begin{aligned}\mathcal{B}(v_P, v_H, v_E) &:= \{ f \in \mathcal{A} : \\ &\quad f = f_P + f_H + f_E, \\ &\quad f_P \in \mathcal{PA}, f_H \in \mathcal{HA}, f_E \in \mathcal{A}, \\ &\quad f_P(X, Y) = \sum_{0 \leq j+k \leq N} f_{jk} e_{jk}(X, Y), f_{jk} \in [b_{jk}, c_{jk}], \\ &\quad \|f_H\| \leq v_H, \|f_E\| \leq v_E \},\end{aligned}$$

Use **directed rounding modes** for **computer-representable numbers** for bounds (v_P, v_H, v_E) where

$$v_P = ([b_{jk}, c_{jk}])_{0 \leq j+k \leq N} \text{ and } v_H, v_E \geq 0.$$

Method of the proof

Newton's Method

We want to find the conjectured 2-cycle $P_1 = (G_1, F_1)$ and $P_2 = (G_2, F_2)$ of the renormalisation operator $\mathcal{T} = \text{diag}(T_1, T_2)$:

$$T_1(G_1, F_1) = (G_2, F_2) \quad \text{and} \quad T_2(G_2, F_2) = (G_1, F_1).$$

We define a new operator (and its derivative) acting on $V = (P_1, P_2)$ using the swapping operator $S(P_1, P_2) = (P_2, P_1)$,

$$\mathcal{F} := I - S\mathcal{T}, \quad D\mathcal{F}(V) = I - SD(\mathcal{T})(V)$$

$$\implies D\mathcal{F}(P_1, P_2)(\delta P_1, \delta P_2) = (\delta P_1 - DT_2(P_2)\delta P_2, \delta P_2 - DT_1(P_1)\delta P_1).$$

We then define the Newton operator

$$\mathcal{N} : V \mapsto V - [D\mathcal{F}(V)]^{-1}[\mathcal{F}(V)].$$

A fixed point V^* of $\mathcal{N}$ is a 2-cycle of T .

We consider the approximate fixed point/2-cycle V^0 and define

$$\Delta \simeq D\mathcal{T}(V^0) \quad \text{and} \quad \Gamma = [I - S\Delta]^{-1}.$$

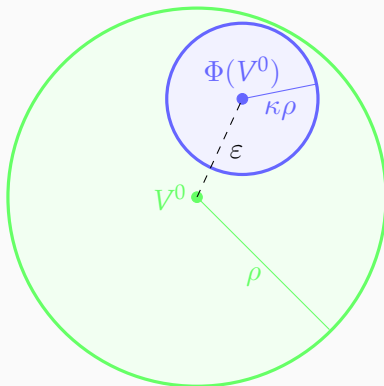
This gives the quasi-Newton operator and its derivative

$$\begin{aligned}\Phi : V &\mapsto V - \Gamma[\mathcal{F}(V)] \\ D\Phi(V) : \delta V &\mapsto \delta V - \Gamma[D\mathcal{F}(V)\delta V].\end{aligned}$$

Quasi-Newton operator as a contraction

We want to show the quasi-Newton operator Φ is a uniform contraction in a ball $B(V^0, \rho)$:

1. $\|\Phi(V^0) - V^0\| \leq \varepsilon$ and select $\rho > \varepsilon$;
2. $\|\Phi(V) - \Phi(U)\| \leq \kappa\|V - U\| \quad \forall U, V \in B(V^0, \rho), \quad \kappa < 1$;
3. Verify that $\varepsilon < \rho(1 - \kappa)$.



Mean value theorem:

$$\|\Phi(W) - \Phi(U)\| \leq \sup_V \|D\Phi(V)\|_{\text{op}} \|W - U\|.$$

Operator norm:

$$\|D\Phi(V)\|_{\text{op}} := \sup_{\|U\|=1} \|D\Phi(V)(U)\| \leq \sup_{1 \leq i \leq 4; j+k \leq N} \|D\Phi(V)(e_{ijk})\|_1.$$

Formal Fréchet derivative $DT(G, F)$

Let $\tilde{G} = G(a^2X, bY)$ and $\tilde{F} = F(a^2X, bY)$. The first component of the Fréchet derivative of T is

$$DT(G, F)(\delta G, \delta F)(X, Y)_1 = -a^{-2}\delta a G(\tilde{G}^2, \tilde{F}) + a^{-1} \left[\delta G(\tilde{G}^2, \tilde{F}) + \partial_1 G(\tilde{G}^2, \tilde{F}) 2\tilde{G} \delta \tilde{G} + \partial_2 G(\tilde{G}^2, \tilde{F}) \delta \tilde{F} \right]$$

(similarly for the second component) where

$$\delta \tilde{G} = \delta G(a^2X, bY) + \partial_1 G(a^2X, bY) 2a \delta a X + \partial_2 G(a^2X, bY) \delta b Y,$$

$$\delta \tilde{F} = \delta F(a^2X, bY) + \partial_1 F(a^2X, bY) 2a \delta a X + \partial_2 F(a^2X, bY) \delta b Y,$$

$$\begin{aligned} \delta a &= \delta G(G(0,0)^2, F(0,0)) + \partial_1 G(G(0,0)^2, F(0,0)) 2G(0,0) \delta G(0,0) \\ &\quad + \partial_2 G(G(0,0)^2, F(0,0)) \delta F(0,0), \end{aligned}$$

$$\begin{aligned} \delta b &= \delta F(G(0,0)^2, F(0,0)) + \partial_1 F(G(0,0)^2, F(0,0)) 2G(0,0) \delta G(0,0) \\ &\quad + \partial_2 F(G(0,0)^2, F(0,0)) \delta F(0,0). \end{aligned}$$

Results

Existence of the C-type two-cycle

Theorem: The conjectured C-type two-cycle

$V^* = ((G_1^*, F_1^*), (G_2^*, F_2^*))$ exists with scaling constants:

$$\alpha_1^* := 1/a_1^* \in [-2.000377729, -2.000377707],$$

$$\beta_1^* := 1/b_1^* \in [-7.646886669, -7.646886254],$$

$$\alpha_2^* := 1/a_2^* \in [-3.282055430, -3.282055360],$$

$$\beta_2^* := 1/b_2^* \in [-2.892710470, -2.892710346],$$

giving, for quadrupling of the period,

$$\alpha^* := \alpha_1^* \alpha_2^* \in [6.565350378, 6.565350586],$$

$$\beta^* := \beta_1^* \beta_2^* \in [22.12022698, 22.12022913].$$

Taking $N = 24$ we compute the bound

$$\|\Phi(V^0) - V^0\|_{\text{sum},1} \leq \varepsilon = 1.5089 \times 10^{-10},$$

and let $\rho = 7.5443 \times 10^{-10} > \varepsilon$. In the ball $B(V^0, \rho)$, we get

$$\|D\Phi(V)\| \leq \kappa = 0.78693 < 1,$$

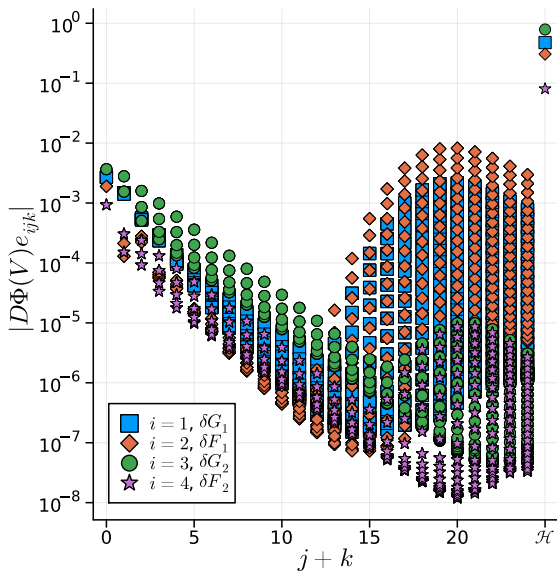
showing that Φ is a uniform contraction. Then we have

$\varepsilon < \rho(1 - \kappa)$ meaning $\Phi(B(V^0, \rho)) \subset B(V^0, \rho)$.

Hence, by the contraction mapping theorem, the 2-cycle

$V^* = (G_1^*, F_1^*, G_2^*, F_2^*,) \in B(V^0, \rho)$ exists.

Upper bounds on norms



Two-cycle vs. fixed points

Find $(Z, W) \in \Omega_1 \cap \Omega_2$ such that $G_1(Z, W) \neq G_2(Z, W)$, valid for all function pairs $(G_1, G_2) \in \mathcal{B}(G_1^0; 0, \rho) \oplus \mathcal{B}(G_2^0; 0, \rho)$. Similarly for $F_1(Z, W)$ and $F_2(Z, W)$.

This is true since the intervals bounding the a_1^* and a_2^* are disjoint:

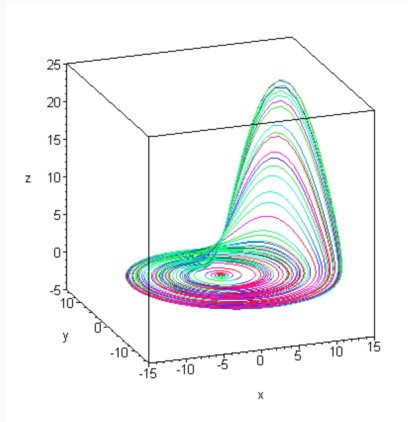
$$a_1^* \in [-0.4999055911, -0.4999055857],$$

$$a_2^* \in [-0.1307721819, -0.1307721747].$$

- Hyperbolicity of the of 2-cycle V^* : Notable eigenvalues of $D(T^2)(G, F)$:
 - $\delta_1 \approx 92.431$,
 - $\delta_2 \approx 4.1924$,
 - $\delta_3 = \beta^* \approx 22.120$,
 - $\lambda \approx 0.93326$.
- FS-type fixed points gives rise to a family of C-type two-cycles: degree d at critical point is a bifurcation parameter and R itself undergoes period-doubling at a critical value $d = d_c < 2$.

Applications

Driven Rössler oscillator at the period-doubling accumulations on the edge of synchronization tongues, and biologically-relevant models of kidney blood flow regulation.



References

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- [J-P Eckmann, H Koch, P Wittwer. *A Computer-Assisted Proof of Universality for Area-Preserving Maps*, Amer. Math. Soc., 1984]
- [Kuznetsov, Kuznetsov, Sataev. *Multiparameter Critical Situations, Universality and Scaling in Two-Dimensional Period-Doubling Maps*, Journal of Statistical Physics volume 121, pp.697–748 (2005)]
- [Laugesen, Mosekilde, Holstein-Rathlou. *C-type period-doubling transition in nephron autoregulation*, 2011 Interface Focus 1:132–142]

THANK YOU.



Feigenbaum Conjectures

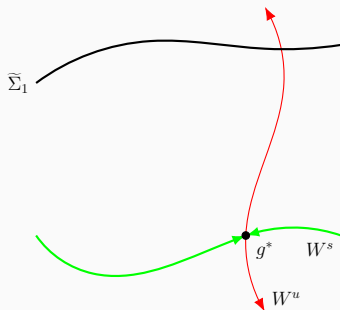


Figure 1: FC1. Existence of a fixed point g^* of R with universal scaling constant $\alpha = 1/a$.

FC2. The fixed point is hyperbolic (saddle-type) with $\delta \in \sigma(DR(g^*))$ essentially the only relevant expanding eigenvalue.

Feigenbaum Conjectures

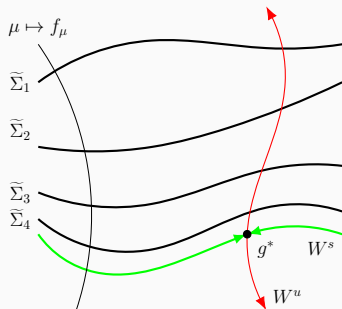


Figure 2: FC3. Transversality: The unstable manifold W^u of g^* crosses the hypersurface $\tilde{\Sigma}_1$ corresponding to maps with superstable period- 2^1 orbit.

Normalisation constants

$$a_1^* \in [-0.4999055911, -0.4999055857],$$

$$b_1^* \in [-0.3046871214, -0.3046871149],$$

$$a_2^* \in [-0.1307721819, -0.1307721747],$$

$$b_2^* \in [-0.3456965546, -0.3456965398],$$

and for period-quadrupling:

$$a^* := a_1^* a_2^* \in [0.1523147906, 0.1523147955],$$

$$b^* := b_1^* b_2^* \in [0.04520748832, 0.04520749271].$$