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Validated polynomial optimisation for computer-assisted proofs in ODEs

Jeremy Parker

Motivating example: Lyapunov functions

Consider an autonomous ODE system

$$\frac{d}{dt}x(t) = f\big(x(t)\big), \qquad f \in C^1(\Omega, \mathbb{R}^n),$$

where

$$\Omega \subset \mathbb{R}^n$$

is forward invariant.

Suppose that $f(0) = 0$. If there exists $V \in C^1(\Omega, \mathbb{R})$ such that

$$\begin{aligned} \mathcal{L}_f V(x) &\leq 0, \qquad && \forall x \in \Omega, \\ V(x) &\geq |x|^2, \qquad && \forall x \in \Omega. \end{aligned}$$

Then the equilibrium $x = 0$ is globally (Lyapunov) stable in Ω.

Here $\mathcal{L}_f V = f \cdot \nabla V$ is the Lie derivative of V with respect to the flow.

Motivating example: Lyapunov functions

Consider an autonomous ODE system

$$\frac{d}{dt}x(t) = f\big(x(t)\big), \qquad x \in \mathbb{R}^n, \qquad f \in \mathbb{R}[x]^n,$$

where

$$\Omega = \{x \in \mathbb{R}^n \mid g(x) \geq 0\}, \qquad g \in \mathbb{R}[x],$$

Multivariate polynomial in x with real coefficients

is forward invariant.

Suppose that $f(0) = 0$. If there exists $V \in \mathbb{R}[x]$ such that

$$-\mathcal{L}_f V = \sigma_0 + g\,\sigma_1, \qquad \sigma_0, \sigma_1 \in SOS,$$
$$V(x) - |x|^2 = \sigma_2 + g\,\sigma_3, \qquad \sigma_2, \sigma_3 \in SOS,$$

Then the equilibrium $x = 0$ is globally (Lyapunov) stable in Ω.

Here $\sigma \in SOS$ means σ is a polynomial sum-of-squares, i.e. $\sigma = \sum p_i^2$ for some $p_i \in \mathbb{R}[x]$.

SOS as SDP

$\sigma = \sum p_i^2$ for some $p_i \in \mathbb{R}[x]$ if and only if

$$\sigma = \mathbf{b}^T Q \mathbf{b}, \qquad Q \succcurlyeq 0,$$

Q is a real symmetric positive semidefinite matrix

where $\mathbf{b}$ is a vector of monomials of sufficiently high degree.

So the problem reduces to finding $V \in \mathbb{R}[x]$ and $Q_1, Q_2, Q_3, Q_4 \succcurlyeq 0$ such that

$$-\mathcal{L}_f V = \mathbf{b}^T Q_1 \mathbf{b} + g\mathbf{b}^T Q_2 \mathbf{b},$$
$$V(x) - |x|^2 = \mathbf{b}^T Q_3 \mathbf{b} + g\mathbf{b}^T Q_4 \mathbf{b}.$$

This is a semidefinite program (SDP). Lots of floating point solvers exist.

Other Lyapunov/auxiliary/storage function methods

SIAM J. Applied Dynamical Systems
Vol. 15, No. 4, pp. 1962–1988

Bounds for Deterministic and Stochastic Dynamical Systems using Sum-of-Squares Optimization*

G. Fantuzzi†, D. Goluskin‡, D. Huang§, and S. I. Chernyshenko†

$$\mathbf{f} \cdot \nabla V + \varphi - L \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

SIAM J. Applied Dynamical Systems
Vol. 19, No. 3, pp. 1823–1864

Bounding Extreme Events in Nonlinear Dynamics Using Convex Optimization*

Giovanni Fantuzzi† and David Goluskin‡

$$\mathcal{L}V(t,x) \leq 0 \quad \forall (t,x) \in \Omega,$$
$$\Phi(t,x) - V(t,x) \leq 0 \quad \forall (t,x) \in \Omega.$$

IOP Publishing | London Mathematical Society Nonlinearity
Nonlinearity 33 (2020) 4878–4899 https://doi.org/10.1088/1361-6544/ab8f7b

Bounding extrema over global attractors using polynomial optimisation

David Goluskin

$$\Phi_A^+ \leqslant \inf_{\substack{\lambda > \mathbb{R} \\ V \in C^1}} C \quad \text{s.t.} \quad \begin{aligned} V(\mathbf{x}) - \Phi(\mathbf{x}) &\geqslant 0 \quad \forall \mathbf{x} \in \mathbb{R}^n, \\ C - V(\mathbf{x}) - \lambda \mathbf{f}(\mathbf{x}) \cdot \nabla V(\mathbf{x}) &\geqslant 0 \quad \forall \mathbf{x} \in \mathbb{R}^n. \end{aligned}$$

IOP Publishing | London Mathematical Society Nonlinearity
Nonlinearity 37 (2024) 095022 (17pp) https://doi.org/10.1088/1361-6544/ad68bb

The Lorenz system as a gradient-like system

Jeremy P Parker

$$f(x) \cdot \nabla V(x) \geqslant \|f(x)\|^2$$

Mathematics > Dynamical Systems

[Submitted on 16 Oct 2025 (v1), last revised 21 Jan 2026 (this version, v2)]

Computation of attractor dimension and maximal sums of Lyapunov exponents using polynomial optimization

Jeremy P Parker, David Goluskin

$$d_L \leq \inf_{\substack{B \in [0,1) \\ V \in C^1}} (k+B) \quad \text{s.t.} \quad \begin{aligned} V(x,w,y^{k+1}) &\geq |w|^2 \\ f \cdot D_x V + p_k^B \cdot D_w V + \left(Df^{[k+1]} y^{k+1}\right) \cdot D_{y^{k+1}} V &\leq 0, \end{aligned}$$

Mathematics > Dynamical Systems

[Submitted on 15 Oct 2025 (v1), last revised 23 Jun 2026 (this version, v2)]

Computation of minimal periods for ordinary differential equations

Jeremy P. Parker, David Goluskin

$$B^a = \inf_{\substack{Q \succ 0 \\ V \in C^1(\Omega)}} B \quad \text{s.t.} \quad Ba^{\mathsf{T}} Q a - (\mathcal{L}_f a)^{\mathsf{T}} Q (\mathcal{L}_f a) + \mathcal{L}_f V \geq 0 \quad \forall x \in \Omega.$$

Other Lyapunov/auxiliary/storage function methods

IFAC PapersOnLine 52-16 (2019) 484–489

Using SOS and Sublevel Set Volume Minimization for Estimation of Forward Reachable Sets

Morgan Jones * Matthew M. Peet **

$$\min_X \{D(X, FR(X_0, f, T, Y, \gamma))\}$$
$$\text{subject to: } X = \{x \in \mathbb{R}^n : V(x,T) \le 1+\gamma\}$$
$$V(x,0) \le 1 \text{ for all } x \in X_0.$$
$$\frac{\partial V}{\partial t}(x,t) + \nabla_x V(x,t)^T f(x,u,w) \le w^T w,$$
$$\text{for all } x \in X_c, t \in [0,T], u \in Y, w \in \mathbb{R}^{m_w}.$$

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 59, NO. 2, FEBRUARY 2014 297

Convex Computation of the Region of Attraction of Polynomial Control Systems

Didier Henrion and Milan Korda

$$\begin{aligned} d^* = \inf \ & \int_X w(x)\,d\lambda(x) \\ \text{s.t.} \ & \mathcal{L}v(t,x,u) \le 0, && \forall (t,x,u) \in [0,T] \times X \times U \\ & w(x) \ge v(0,x) + 1, && \forall x \in X \\ & v(T,x) \ge 0, && \forall x \in X_T \\ & w(x) \ge 0, && \forall x \in X \end{aligned} \tag{16}$$

Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008
ThB10.1

Nonlinear optimal control synthesis via occupation measures

Didier Henrion, Jean B. Lasserre and Carlo Savorgnan

$$\frac{\partial \varphi(x,t)}{\partial t} + \nabla_x \varphi(x,t) f(t,x,u) + h(t,x,u) \ge 0 \tag{13}$$

for all $(t,x,u) \in \mathcal{C}_T$ and

$$H(x) - \varphi(T,x) \ge 0 \qquad \forall x \in \mathcal{C}_F. \tag{14}$$

Periodic solutions to ODEs

Consider an autonomous ODE system

$$\frac{d}{dt}x(t) = f\big(x(t)\big), \qquad f \in C^1(\Omega, \mathbb{R}^n), \qquad \Omega \subset \mathbb{R}^n$$

A solution $x(t)$ is a periodic orbit/limit cycle if for some $T > 0$ we have $x(t+T) = x(t)$ for all t but $x(t)$ is **not constant.**

Then we call T a period of the system.

Lower bounds for periods: intuition

Theorem (Fenchel 1929)

The total absolute curvature of a smooth closed curve is at least 2π.

- The local curvature and speed of trajectories is limited, so the periods should be bounded below, at least locally.

Lower bounds for periods: existing results

Theorem (Yorke 1969)

If f is Lipschitz continuous with constant L, i.e. $|f(y) - f(x)| \leq L|y - x|$ for all $x, y \in \Omega$, then periods satisfy

$$T \geq \frac{2\pi}{L}.$$

- This holds for all Hilbert spaces. For Banach spaces $T \geq 6/L$.
- More results for different classes of (infinite dimensional) system.
- This is a sharp bound, e.g. linear oscillator.
- This is not sharp in general for a particular system.
- In fact, f is not generally Lipschitz, but we typically still expect periods to be bounded below.

A lemma

Theorem (Wirtinger's inequality)

If $f \in C^1([0,T],\mathbb{R}^m)$ satisfies $f(T) = f(0)$ and $\int_0^T f(t)\,dt = 0$, then

$$\int_0^T |f(t)|^2\,dt \le \left(\frac{T}{2\pi}\right)^2 \int_0^T |f'(t)|^2\,dt\,.$$

- Poincaré inequality on the circle
- Equality if the components of f are sinusoidal with period T.

Sketch proof

Theorem (Yorke 1969)
If f is Lipschitz continuous with constant L then periods satisfy
$$T \geq \frac{2\pi}{L}.$$

Proof.
Consider any periodic orbit with period T.
$f\big(x(t)\big)$ satisfies conditions for Wirtinger's inequality, so
$$\left(\frac{2\pi}{T}\right)^2 \int_0^T \left|f\big(x(t)\big)\right|^2 dt \leq \int_0^T \left|\mathcal{L}_f f\big(x(t)\big)\right|^2 dt\,.$$
Also $\left|\mathcal{L}_f\, f(x)\right| \leq L|f(x)|$, so
$$\int_0^T \left|\mathcal{L}_f\, f\big(x(t)\big)\right|^2 dt \leq L^2 \int_0^T \left|f\big(x(t)\big)\right|^2 dt\,.$$

Generalising

Proposition 1

Suppose there exist $B > 0, V \in C^1(\Omega, \mathbb{R})$ and $\Phi \in C^1(\Omega, \mathbb{R}^m)$ such that

$$B\,|\Phi(x)|^2 - \left|\mathcal{L}_f \Phi(x)\right|^2 + \mathcal{L}_f V(x) \geq 0, \qquad \forall x \in \Omega.$$

Suppose $x(t)$ is T-periodic for some $T > 0$. If $\Phi\big(x(t)\big)$ satisfies the mean-zero condition $\int_0^T \Phi\big(x(t)\big)dt = 0$ but does not vanish everywhere on $x(t)$, then

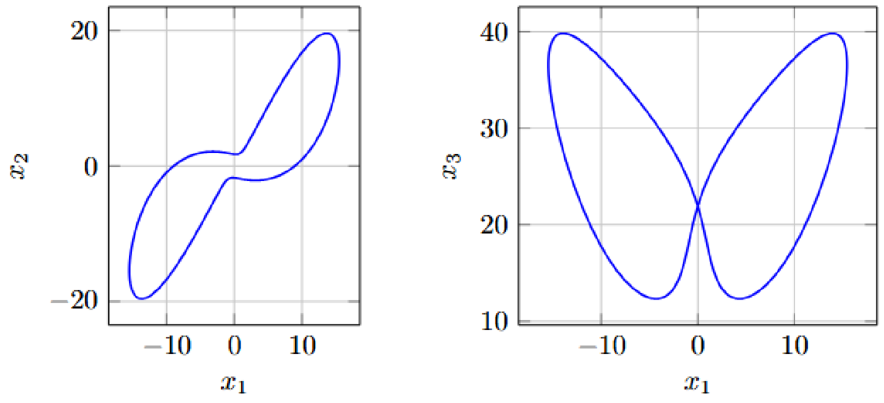
$$T \geq \frac{2\pi}{\sqrt{B}}.$$

- This reduces to the Yorke result when $\Phi = f, V = 0$ and $B = L^2$.

Example: Lorenz '63

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \sigma\left(x_2 - x_1\right),$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1\left(\rho - x_3\right) - x_2,$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1 x_2 - \beta x_3.$$

$$\beta = \frac{8}{3}, \sigma = 10, \rho = 28$$



Shortest known orbit with $T \approx 1.5587$

Many orbits are invariant under $x \mapsto (-x_1, -x_2, x_3)$.

In that case, x_1 must integrate to zero, and $x_1 \equiv 0$ iff the trajectory tends to the origin.

Example: Lorenz '63

Proposition 1

Suppose there exist $B > 0, V \in C^1(\Omega, \mathbb{R})$ and $\Phi \in C^1(\Omega, \mathbb{R}^m)$ such that

$$B\,|\Phi(x)|^2 - \left|\mathcal{L}_f \Phi(x)\right|^2 + \mathcal{L}_f V(x) \geq 0, \qquad \forall x \in \Omega.$$

Suppose $x(t)$ is T-periodic for some $T > 0$. If $\Phi\big(x(t)\big)$ satisfies the mean-zero condition $\int_0^T \Phi\big(x(t)\big)dt = 0$ but does not vanish everywhere on $x(t)$, then

$$T \geq \frac{2\pi}{\sqrt{B}}.$$

Take $\Phi(x) = x_1, B = \sigma^2(\rho - 1), V = \frac{1}{2}(\sigma(\rho - 2)x_1^2 - \sigma^2 x_2^2 - \sigma^2 x_3^2)$, then

$$B\,|\Phi(x)|^2 - \left|\mathcal{L}_f \Phi(x)\right|^2 + \mathcal{L}_f V(x) = \beta\sigma^2 x_3^2$$

so symmetric orbits have

$$T \geq \frac{2\pi}{\sigma\sqrt{\rho - 1}} \approx 0.1209 \ll 1.5587$$

Relaxing to an SDP

Proposition 1

Suppose there exist $B > 0, V \in C^1(\Omega, \mathbb{R})$ and $\Phi \in C^1(\Omega, \mathbb{R}^m)$ such that

$$B\,|\Phi(x)|^2 - \left|\mathcal{L}_f \Phi(x)\right|^2 + \mathcal{L}_f V(x) \geq 0, \qquad \forall x \in \Omega.$$

Suppose $x(t)$ is T-periodic for some $T > 0$. If $\Phi\big(x(t)\big)$ satisfies the mean-zero condition $\int_0^T \Phi\big(x(t)\big)dt = 0$ but does not vanish everywhere on $x(t)$, then

$$T \geq \frac{2\pi}{\sqrt{B}}.$$

Key idea: choose $\Phi = Ua$ for some a, $Q = U^T U$

Proposition 2

Suppose there exist $B > 0, V \in C^1(\Omega, \mathbb{R})$, $a \in C^1(\Omega, \mathbb{R}^m)$ and $Q \succ 0$ such that

$$B\, a^T Q a - \big(\mathcal{L}_f a\big)^T Q \big(\mathcal{L}_f a\big) + \mathcal{L}_f V(x) \geq 0, \qquad \forall x \in \Omega.$$

Suppose $x(t)$ is T-periodic for some $T > 0$. If $a\big(x(t)\big)$ satisfies the mean-zero condition $\int_0^T a\big(x(t)\big)dt = 0$ but does not vanish everywhere on $x(t)$, then

$$T \geq \frac{2\pi}{\sqrt{B}}.$$

Relaxing to an SDP

Proposition 2

Suppose there exist $B > 0, V \in C^1(\Omega, \mathbb{R}), a \in C^1(\Omega, \mathbb{R}^m)$ and $Q \succ 0$ such that

$$B\, a^T Q a - \left(\mathcal{L}_f a\right)^T Q\left(\mathcal{L}_f a\right) + \mathcal{L}_f V(x) \geq 0, \qquad \forall x \in \Omega.$$

Suppose $x(t)$ is T-periodic for some $T > 0$. If $a\big(x(t)\big)$ satisfies the mean-zero condition $\int_0^T a\big(x(t)\big)dt = 0$ but does not vanish everywhere on $x(t)$, then

$$T \geq \frac{2\pi}{\sqrt{B}}.$$

Proposition 3

Suppose $f \in \mathbb{R}^n[x]$ and there exist $B > 0, V \in \mathbb{R}[x], w \in \mathbb{R}^m[x]$ and $Q \succ 0$ such that

$$B\, a^T Q a - \left(\mathcal{L}_f a\right)^T Q\left(\mathcal{L}_f a\right) + \mathcal{L}_f V(x) \geq 0, \qquad \forall x \in \Omega,$$

where $a = L_f w$, and suppose also that the n components of x are among the m components of w.

Then all periods are bounded by

$$T \geq \frac{2\pi}{\sqrt{B}}.$$

Concretely

Suppose

$$\Omega = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \text{ for } i = 1, \dots, I,\ h_j(x) = 0 \text{ for } j = 1, \dots, J\},$$

Choose bases $a = \mathcal{L}_f w$, b_i, c, c_j. Let $\Phi = Ua$ and $Q = U^T U$. All periods T are bounded below via

$$\left(\frac{2\pi}{T}\right)^2 \leq \inf_{\substack{\mathsf{Q} \in \mathbb{S}^m \\ \mathsf{P}_i \in \mathbb{S}^{m_i} \\ v \in \mathbb{R}^{n_0} \\ r_j \in \mathbb{R}^{n_j}}} B \quad s.t. \quad \begin{aligned} & B a^\mathsf{T} \mathsf{Q} a - \left(\mathcal{L}_f^2 a\right)^\mathsf{T} \mathsf{Q} \left(\mathcal{L}_f^2 a\right) + v^\mathsf{T} \mathcal{L}_f c = b_0^\mathsf{T} \mathsf{P}_0 b_0 + \sum_{i=1}^{I} b_i^\mathsf{T} \mathsf{P}_i b_i g_i + \sum_{j=1}^{J} r_j^\mathsf{T} c_j h_j. \\ & \mathsf{Q} \succ 0 \text{ and all } \mathsf{P}_i \succeq 0. \end{aligned}$$

Validation

$$Ba^{\mathsf{T}}\mathsf{Q}a - \left(\mathcal{L}_f^2 a\right)^{\mathsf{T}}\mathsf{Q}\left(\mathcal{L}_f^2 a\right) + v^{\mathsf{T}}\mathcal{L}_f c = b_0^{\mathsf{T}}\mathsf{P}_0 b_0 + \sum_{i=1}^{I} b_i^{\mathsf{T}}\mathsf{P}_i b_i g_i + \sum_{j=1}^{J} r_j^{\mathsf{T}} c_j h_j. \qquad (20)$$

Algorithm 1 Verified computation of lower bounds on periods using rational arithmetic.

Inputs: $f \in \mathbb{R}^n[x]$, $a \in \mathbb{R}^m[x]$, $c \in \mathbb{R}^{n_0}[x]$, $b_i \in \mathbb{R}^{m_i}[x]$ for $i = 0, \dots, I$, and $c_j \in \mathbb{R}^{n_j}[x]$ for $j = 1, \dots, J$, all with integer coefficients.

Output: $B \in \mathbb{Q}$ such that $\inf_{\mathcal{P}} T \geq \frac{2\pi}{\sqrt{B}}$.

1. Fix a first value of $B \in \mathbb{Q}$.
2. Attempt to solve the SDP feasibility problem (20), using floating point arithmetic and with the constraint $tr\,\mathsf{Q} = 1$ included to fix the variables' magnitudes.
3. If the SDP solution fails to converge, or if it returns a solution that does not satisfy the constraints within a specified tolerance, increase B and repeat step 2. If the SDP solution returns floating point variables $\mathsf{Q} \in \mathbb{S}^m$, $\mathsf{P}_i \in \mathbb{S}^{m_i}$, $v \in \mathbb{R}^{n_0}$ and $r_j \in \mathbb{R}^{n_j}$ that nearly satisfy the constraints, including all matrices being positive definite, proceed with rational validation.
4. Express the equality constraint in (20) as a linear relation $\mathsf{A}y = 0$, where the vector y contains all variables in Q, P_i, v and r_j. Choose A to have integer entries, which may require multiplying all terms by the denominator of B.
5. Use rational Gaussian elimination to construct a matrix K whose columns are an exact basis for the kernel of A.
6. Using the floating point result from step 2, construct a rational vector y_0 that approximately solves $\mathsf{A}y = 0$.
7. Find a floating point vector x_0 minimizing $|\mathsf{K}x_0 - y_0|$ using the LSQR algorithm.
8. Approximate x_0 as a rational vector x, so that $y = \mathsf{K}x$ exactly solves $\mathsf{A}y = 0$.
9. Build rational Q, P_i, v and r_j from y that satisfy the equality constraint in (20) by construction.
10. Validate that the rational matrices satisfy $\mathsf{Q} \succ 0$ and $\mathsf{P}_i \succeq 0$. This is done via $\mathsf{LDL}^{\mathsf{T}}$ decomposition, which is exact with rational arithmetic.
11. If validation fails at any step, increase B and return to step 2. If validation succeeds, decrease B and return to step 2. Carry out a line search for the smallest B value that can be validated, terminating when reaching the desired precision in B.

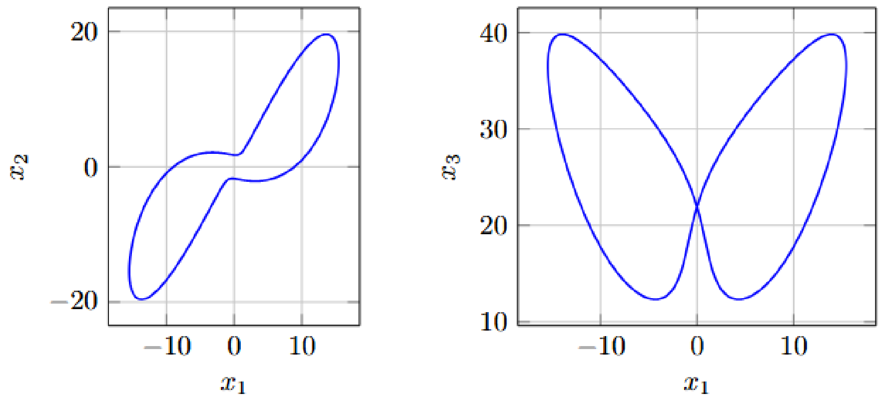
Example: Lorenz ‘63

Choose w, b and c to be monomials* up to degree d_a, d_b and d_c, and $a = \mathcal{L}_f w$.

$$\Omega = \mathbb{R}^3$$

$$Ba^{\mathsf{T}}\mathsf{Q}a - \left(\mathcal{L}_f^2 a\right)^{\mathsf{T}}\mathsf{Q}\left(\mathcal{L}_f^2 a\right) + v^{\mathsf{T}}\mathcal{L}_f c = b_0^{\mathsf{T}}\mathsf{P}_0 b_0$$

d_a	d_b	d_c	B	Lower bound on period	SDP time (s)	Validation time (s)
3	3	6	1128	**0.5612**	0.06	0.03
4	4	8	888	**0.6325**	0.44	0.62
5	5	10	488	**0.8532**	4.01	8.98
6	6	12	325	**1.0455**	29.2	79.7
7	7	14	155.9	**1.5096**	81.5	573
8	8	16	146.26	**1.5586**	641	3558



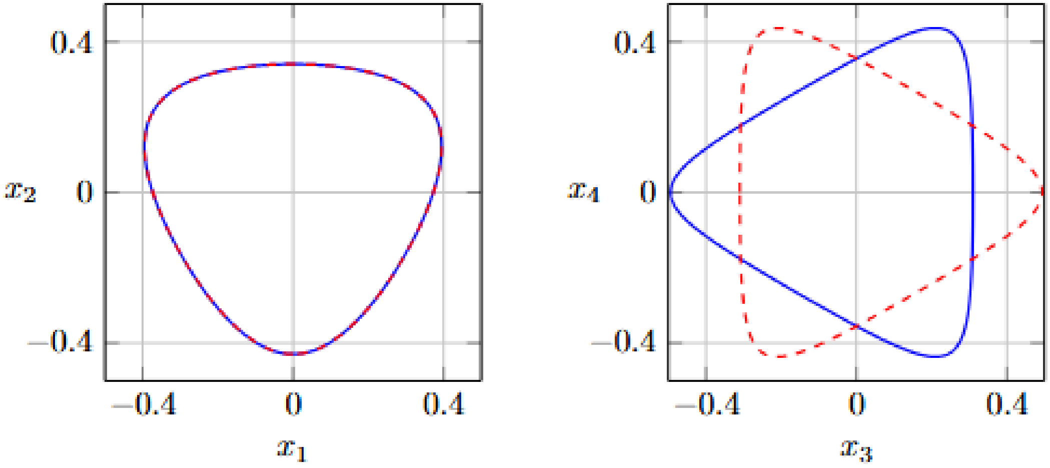
Shortest known orbit with $T \approx 1.55865$

Example: Hénon-Heiles

$$H(x) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) + x_1^2 x_2 - \frac{1}{3}x_2^3, \qquad f(x) = \left(x_3,\, x_4,\, -x_1 - 2x_1x_2,\, -x_2 - x_1^2 + x_2^2\right).$$

$$\Omega = \{x \in \mathbb{R}^3 \mid 1 - 6H(x) \geq 0\},$$

d_a	d_b	d_c	B	Lower bound on period	SDP time (s)	Validation time (s)
2	3	5	1.28572	**5.5412**	0.33	0.01
2	4	7	1.25210	**5.6151**	7.95	0.38
3	4	7	1.08846	**6.0224**	8.59	0.49



Shortest known orbits with $T \approx 6.02248$

Summary

- SOS optimisation gives computational global results for polynomial ODE systems
- Can be validated with rational arithmetic to give a computer assisted proof
- We extended SOS for bounds on periods

Things I didn’t discuss

- Terrible scaling
- Non-polynomial systems
- Analogous method for polynomial maps
- Extending this to PDEs
- Interval arithmetic validation