Computer Assisted Proofs for Nonlinear Problems in the Borel Plane and Validated Numerics for Complex Convolutions

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Overview:

- In this talk we will look at validated numerical methods for computing complex convolutions.

- Such convolutions appear when using the Borel Transform to study functional equations that have solutions which are C^∞ but not analytic as the Borel Transform turns products into convolutions. These include parabolic center manifolds, solns. of heat type equations, Painlevé equations, ...

- As an illustrative example, we will focus on a "quadratic" Euler equation:

$$x^2y' + y + \varepsilon y^2 = x$$

- When the Borel Transform is applied to this equation we obtain the following:

$$zu(z) + u(z) + \varepsilon \int_0^z u(w)u(z-w)dw = 1$$

Borel Resummation

Strategy for Borel Resummation:

Step 0: Move singularity to infinity by doing a change of variables.

Step 1: Take the Borel Transform of the equation.

Step 2: Study solution at 0.

Step 3: Analytically continue the solution to a set containing $\mathbb{R}^+$.

Step 4: Compute the Laplace Transform.

An illustrative Example of Euler:

Consider the ODE

$$x^2 y' + y = x$$

- Change variables $x \to \frac{1}{t}$

  Obtain

$$-\frac{dy}{dt} + y = \frac{1}{t}$$

$$\mathcal{L}[t^n f] = (-1)^n F^{(n)}(s) \quad \text{and} \quad \mathcal{L}[1] = \frac{1}{s}$$

$$\implies \quad \mathcal{B}[-y'(t)] = z\mathcal{B}[y] \quad \text{and} \quad \mathcal{B}\left[\frac{1}{t}\right] = 1$$

Letting $\mathcal{B}[y] = u(z)$

$$zu(z) + u(z) = 1$$

has solution

$$u(z) = \sum_{n=0}^{\infty} (-1)^n z^n \qquad \text{analytic on } |z| < 1$$

Strategy for Borel resummation:

- Step zero: change variables so singularity is at infinity.
- Borel transform your functional equation
- Study the (analytic) solution at the origin on $\mathbb{C}$
- Analytically continue the solution to a set which contains the positive real axis.
- Compute the Laplace transform.

- Analytic continuation of $u(z)$?

$$u(z)(1+z) = 1$$

or

$$u(z) = \frac{1}{1+z} \qquad z \neq -1$$

- Since $\mathbb{C}\backslash\{-1\}$ contains $[0, \infty)$ we define

$$y(t) = \mathcal{L}[u(z)] = \int_0^{\infty} \frac{1}{1+z} e^{-tz}\, dz$$

An illustrative Example of Euler:

Consider the ODE

$$x^2 y' + y = x$$

Does it have a bounded solution at $x = 0$?

$$y(x) \approx y^N(x) = \sum_{n=1}^{N} (-1)^{n-1}(n-1)!x^n$$

$$|y(x) - y^N(x)| \leq (N+1)!x^{N+2}$$

Strategy for Borel resummation:

- Step zero: change variables so singularity is at infinity.
- Borel transform your functional equation
- Study the (analytic) solution at the origin on $\mathbb{C}$
- Analytically continue the solution to a set which contains the positive real axis.
- Compute the Laplace transform.

$$x = 0.1 \quad |y(0.1) - y^7(0.1)| \leq 4.032 \times 10^{-5}$$

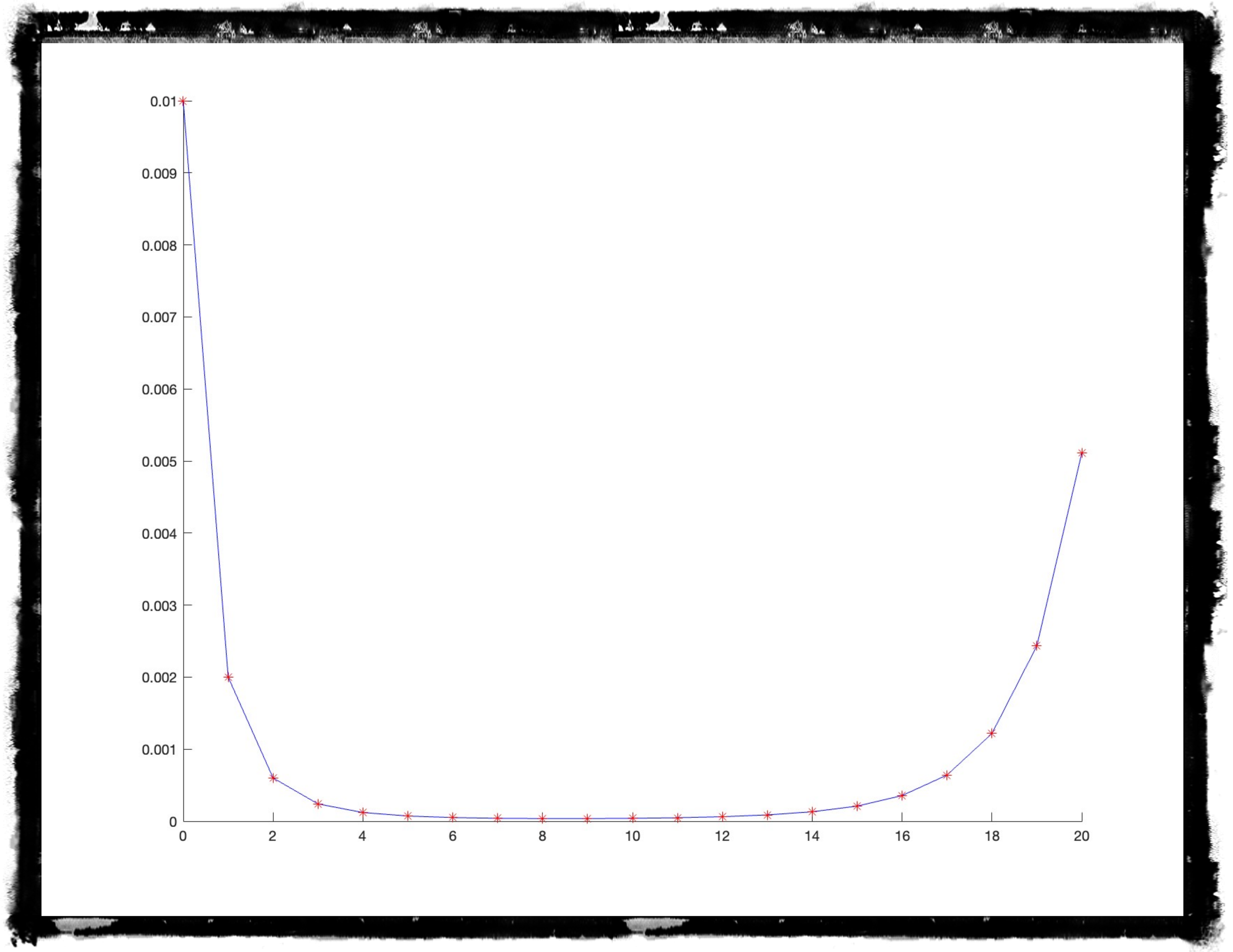
$$|y(0.1) - y^8(0.1)| \leq 3.6288 \times 10^{-5}$$

$$|y(0.1) - y^9(0.1)| \leq 3.6288 \times 10^{-5}$$

$$|y(0.1) - y^{10}(0.1)| \leq 3.99168 \times 10^{-5}$$

$$x = 0.05 \quad |y(0.05) - y^{19}(0.05)| \leq 1.2 \times 10^{-9}$$

$$x = 0.025 \quad |y(0.025) - y^{39}(0.025)| \leq 1.69 \times 10^{-18}$$



Dealing with Convolutions

Modification of Euler's

- We now consider Euler's equation with an extra quadratic term:

$$x^2y' + y + \varepsilon y^2 = x$$

- The term y^2 it's introduces a convolution once we take the Borel transform, since in general we have:

$$\mathcal{L}[f * g] = \mathcal{L}[f]\mathcal{L}[g], \text{ where } (f * g)(z) := \int_0^z f(z-w)g(w)dw$$

- For $\varepsilon = 0$, we have Euler's equation $x^2y' + y = x$ which does not have a quadratic term, hence the equation won't contain a convolution once it's transform.

Modification of Euler's (on $\mathbb{D}_0$)

Step 0: Change of variables

- Apply change of variables to send singularity to infinity by setting $x = \dfrac{1}{t}$

  Then, the equation turns into:

$$-\frac{dy}{dt} + y + \varepsilon y^2 = \frac{1}{t}$$

Step 1: Borel Transform

- Take Borel Transform of equation.

  Recall:

$$\mathcal{L}[t^n f] = (-1)^n F^{(n)}(s) \quad \text{and} \quad \mathcal{L}[1] = \frac{1}{s}$$

$$\implies \quad \mathcal{B}[-y'(t)] = z\mathcal{B}[y] \quad \text{and} \quad \mathcal{B}\left[\frac{1}{t}\right] = 1$$

So, the equation is transformed to:

$$z\mathcal{B}[y] + \mathcal{B}[y] + \varepsilon\mathcal{B}[y] * \mathcal{B}[y] = 1 \qquad \text{OR} \qquad zu(z) + u(z) + \varepsilon \int_0^z u(w)u(z-w)dw = 1, \text{ setting } B[y] = u(z)$$

Modification of Euler's (on $\mathbb{D}_0$)

Step 2: Solution at 0

Matching powers bellow:

$$\sum_{n=0}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} a_n z^n + \varepsilon \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_{n-k} a_k K_{n,k} \right) z^{n+1} = 1$$

Yields:

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ -a_{n-1} - \varepsilon \sum_{k=0}^{n-1} a_{n-k-1} a_k K_{n-1,k} & \text{if } n \geq 1 \end{cases}$$

where: $K_{n,k} := \int_0^1 s^{n-k}(1-s)^k \, ds = B(n-k+1, k+1).$

So our goal is to obtain the error bounds for

$$|y(t) - y^N(t)| = |\mathcal{L}[\bar{u}] - \mathcal{L}[u^N]| = \left| \int_0^{\infty} \bar{u}(z)\, e^{-tz}\, dz - \int_0^{\infty} u^N(z)\, e^{-tz}\, dz \right| = \left| \int_0^{\infty} (\bar{u}(z) - u^N(z))\, e^{-tz}\, dz \right| \leq \quad ?$$

A Fixed Point Problem

Theorem

Let $\mathcal{X}$ be a Banach space, $\bar{x} \in \mathcal{X}$, and $T : \mathcal{X} \to \mathcal{X}$ a continuously differentiable map. Assume there exists a constant Y and a non-decreasing map $Z : [0, \infty) \to [0, \infty)$ such that

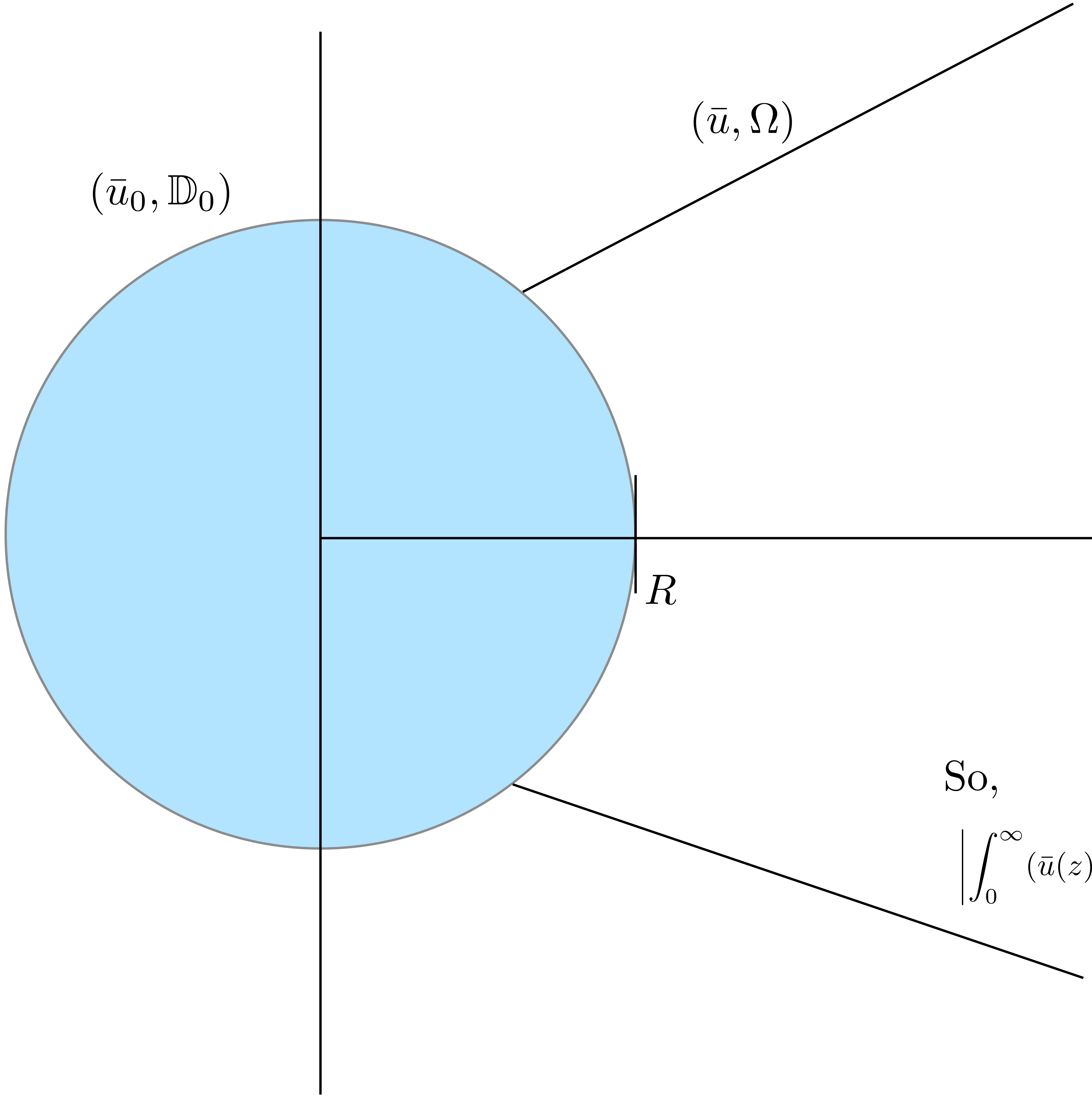
$$\begin{aligned} \|T(\bar{x}) - \bar{x}\| &\leq Y, \\ \|DT(x)\| &\leq Z(\|x - \bar{x}\|), \quad \forall x \in \mathcal{X}. \end{aligned}$$

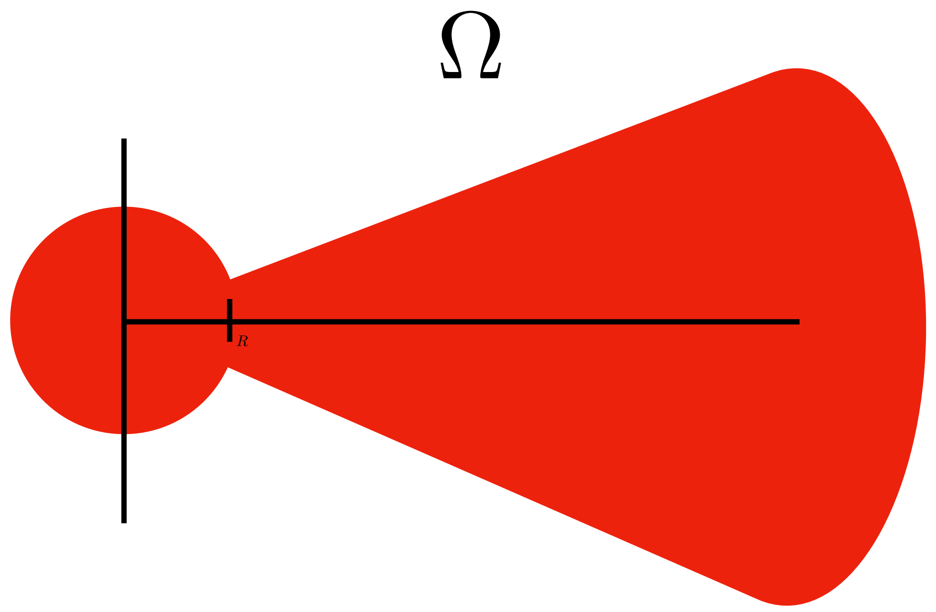
If there exists $r > 0$ such that

$$\begin{aligned} Y + \int_0^r Z(s)\mathrm{d}s &\leq r, \\ Z(r) &< 1, \end{aligned}$$

then T has a unique fixed point x^* such that $\|x^* - \bar{x}\| \leq r$.

We make use of this Newton-Kantorovich type theorem, twice. One to obtain control of the tail in $\mathbb{D}_0$, and another for our continuation argument.





So,

$$\left| \int_0^\infty (\bar{u}(z) - u^N(z))\, e^{-tz}\, dz \right| = \left| \int_0^R (\bar{u}_0(z) - u^N(z))\, e^{-tz}\, dz + \int_R^\infty (\bar{u}(z) - u^N(z))\, e^{-tz}\, dz \right|$$
$$\leq \left| \int_0^R (\bar{u}_0(z) - u^N(z))\, e^{-tz}\, dz \right| + \left| \int_R^\infty (\bar{u}(z) - u^N(z))\, e^{-tz}\, dz \right|$$

A Fixed Point Problem

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Define T as

$$T_0 = a_0, \quad T_n(a_0, \ldots, a_{n-1}) = a_n.$$

And S, δ_i

$$S(a) = \begin{cases} 0 & \text{if } n = 0 \\ a_{n-1} & \text{if } n \geq 1 \end{cases} \qquad \delta_i = \begin{cases} 1 & \text{if } n = i \\ 0 & \text{if } n \neq i \end{cases}$$

So T becomes

$$T(a) = -S(a) - \varepsilon S(a \star a) + a \cdot \delta_0$$

Here, we are interested in obtaining a bound on a^∞ such that

$$T^\infty(a^\infty) = \pi^\infty T(a^N + a^\infty) = a^\infty$$

A Fixed Point Problem

Note: The R here is the radius of $\mathbb{D}_0$.

Theorem

Let $\mathcal{X}$ be a Banach space, $\bar{x} \in \mathcal{X}$, and $T : \mathcal{X} \to \mathcal{X}$ a continuously differentiable map. Assume there exists a constant Y and a non-decreasing map $Z : [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned} \|T(\bar{x}) - \bar{x}\| &\le Y, \\ \|DT(x)\| &\le Z(\|x - \bar{x}\|), \quad \forall x \in \mathcal{X}. \end{aligned}$$

If there exists $r > 0$ such that

$$\begin{aligned} Y + \int_0^r Z(s)\mathrm{d}s &\le r, \\ Z(r) &< 1, \end{aligned}$$

then T has a unique fixed point x^* such that $\|x^* - \bar{x}\| \le r$.

Consider T^∞ the space $\ell^{1,\infty}_R$ with norm $\|\alpha\|_{1,R} = \sum_{n=0}^{\infty} |\alpha_n| R^n$. And note that T^∞ can be written out as:

$$T^\infty(\alpha) = \pi^\infty T(a^N + \alpha) = -a_N \delta_{N+1} - \pi^\infty S(\alpha) - \varepsilon \pi^\infty S(a^N \star a^N) - 2\varepsilon \pi^\infty S(a^N \star \alpha) - \varepsilon \pi^\infty S(\alpha \star \alpha)$$

So, we have unique a^∞ such that $\|a^\infty\|_{1,R} \boxed{\le r_0}$ if

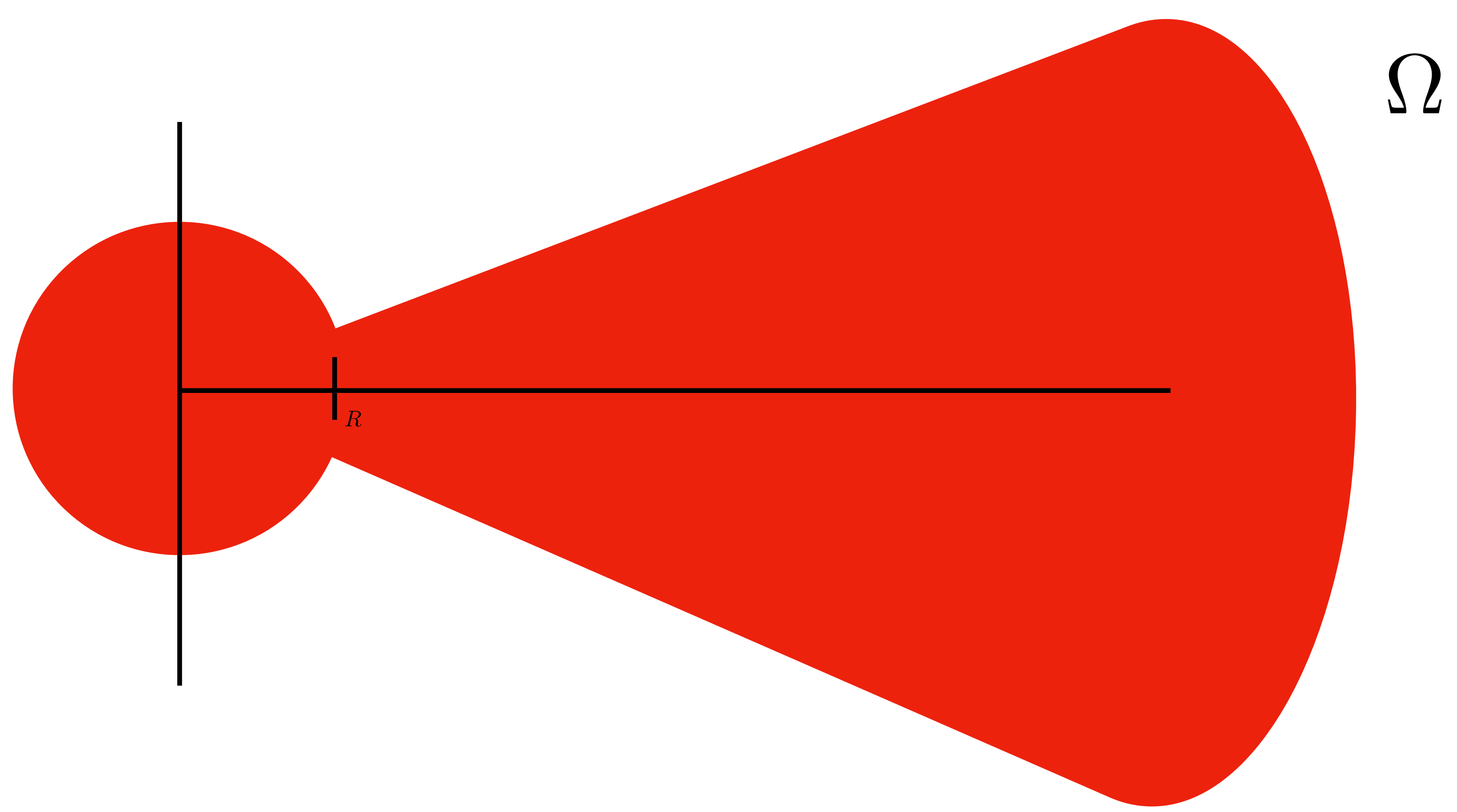
$$Y + \int_0^r Z(s)\, ds = Y + R\left(1 + \frac{2\varepsilon}{N+2}\|a^N\|_{1,R}\right) r_0 + \frac{\varepsilon R}{N+2} r_0^2 \le r_0$$

$$Y = \|a_{N+1}\delta_{N+1} + \varepsilon \pi^\infty S(a^N \star a^N)\|_{1,R}$$

$$\begin{aligned} \|DT^\infty\| &= \sup_{\|b\|_{1,R}=1} \|DT^\infty(\alpha)(b)\|_{1,R} \\ &\le \|\pi^\infty\| \|S\| + \frac{2\varepsilon R}{N+2}\|a^N\|_{1,R} + \frac{2\varepsilon R}{N+2}\|\alpha\|_{1,R} \\ &\le R + \frac{2\varepsilon R}{N+2}(\|a^N\|_{1,R} + \|\alpha\|_{1,R}) \\ &= Z_0(\|\alpha\|_{1,R}) \end{aligned}$$

A Fixed Point Problem

Continuation onto the real axis:



Define the operator H by

$$H[u](z) := \frac{1}{1+z}\left(1 - \varepsilon \int_0^z u(w)u(z-w)\,dw\right)$$

Consider the following space with the defined norm $\|h\|_{\max} = \max\{\|h\|_{\ell^1_R}, \|h\|_\nu\}$

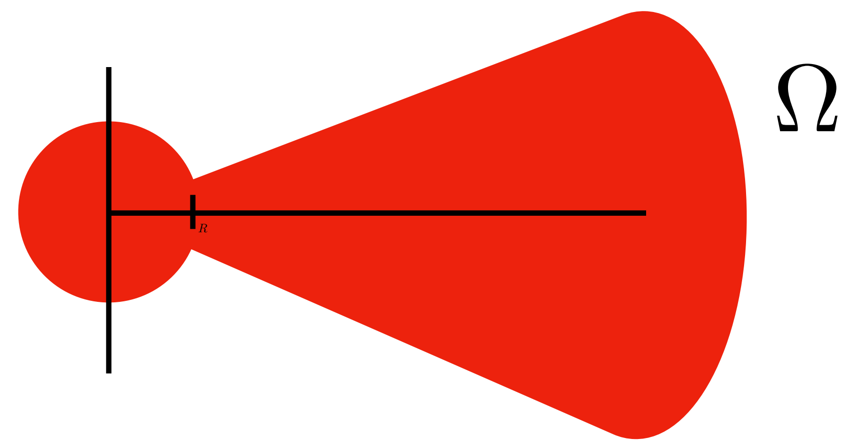
$$X := \left\{h \in \mathcal{A}(\Omega) \mid h^{(n)}(0) = 0,\ n = 0, \ldots, N, \|h\|_{\max} < \infty\right\}$$

$$\|h\|_{\ell^1_R} = \sum_{n=N+1}^{\infty} \left|\frac{h^{(n)}(0)}{n!}\right| R^n, \qquad \|h\|_\nu = \sup_{z\in\Omega} |h(z)|(1+\nu^2|z|^2)e^{-\nu|z|}$$

“Bizarro norm”

We will do a NK type argument on the operator $\hat{H} : X \to X$ $\qquad \hat{H}[h] = H[\bar{u}^N + h] - \bar{u}^N$

A Fixed Point Problem



Why the "Bizarro norm"?

$$\|h\|_\nu = \sup_{z\in\Omega} |h(z)|(1+\nu^2|z|^2)e^{-\nu|z|} \qquad \text{(P. Bonckaert, P. De Maesschalck 2007)}$$

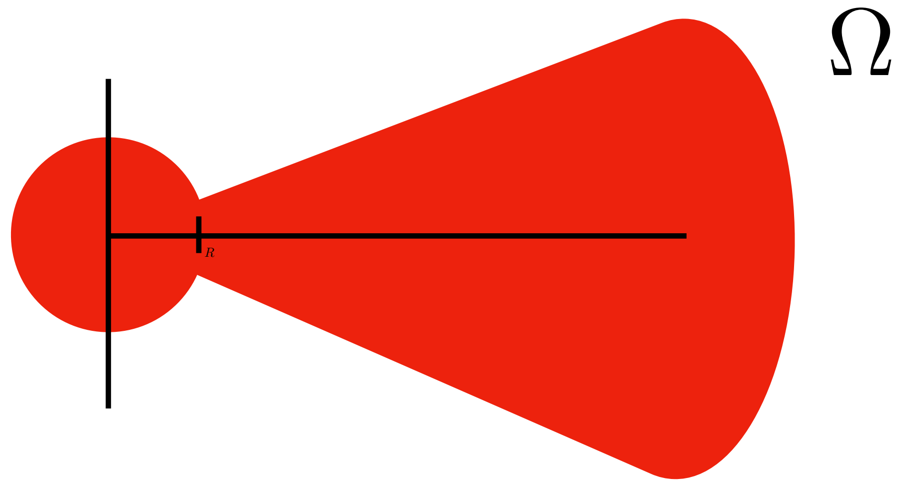
$$\begin{aligned}
|(u*v)(z)| &= \left|\int_0^z u(w)v(z-w)\,dw\right| \\
&= \left|\int_0^1 u(zt)v(z-zt)\cdot z\,dt\right| \\
&\le |z|\int_0^1 \frac{\|u\|_\nu e^{\nu|zt|}}{1+\nu^2|zt|^2}\cdot\frac{\|v\|_\nu e^{\nu|z-zt|}}{1+\nu^2|z-zt|^2}\,dt \\
&\le |z|e^{\nu|z|}\|u\|_\nu\|v\|_\nu\int_0^1 \frac{1}{(1+\nu^2|zt|^2)(1+\nu^2|z-zt|^2)}\,dt \\
&= 2|z|e^{\nu|z|}\|u\|_\nu\|v\|_\nu\int_0^{1/2} \frac{1}{(1+\nu^2(zt)^2)(1+\nu^2|z|^2(1-t)^2)}\,dt \\
&\le 2|z|e^{\nu|z|}\|u\|_\nu\|v\|_\nu\cdot\frac{1}{1+\nu^2(1/4|z|)^2}\int_0^{1/2}\frac{1}{1+\nu^2(zt)^2}\,dt
\end{aligned}$$

$$\begin{aligned}
&\le |z|e^{\nu|z|}\|u\|_\nu\|v\|_\nu\cdot\frac{8}{4+\nu^2|z|^2}\int_0^{1/2}\frac{1}{1+\nu^2(zt)^2}\,dt \\
&= e^{\nu|z|}\|u\|_\nu\|v\|_\nu\cdot\frac{8}{\nu(4+\nu^2|z|^2)}\arctan\left(\frac{\nu|z|}{2}\right) \\
&\le \frac{4\pi}{\nu}\|u\|_\nu\|v\|_\nu\frac{e^{\nu|z|}}{\nu(4+\nu^2|z|^2)}
\end{aligned}$$

$$\Longrightarrow \qquad \|u*v\|_\nu \le \frac{4\pi}{\nu}\|u\|_\nu\|v\|_\nu$$

A Fixed Point Problem

$$X := \left\{ h \in \mathcal{A}(\Omega) \mid h^{(n)}(0) = 0,\ n = 0, \dots, N, \|h\|_{\max} < \infty \right\}$$



Continuation onto the real axis:

$$\|h\|_{\ell^1_R} = \sum_{n=N+1}^{\infty} \left| \frac{h^{(n)}(0)}{n!} \right| R^n, \qquad \|h\|_\nu = \sup_{z \in \Omega} |h(z)|(1 + \nu^2 |z|^2) e^{-\nu |z|}$$

$$\hat{H}[h] = H[\bar{u}^N + h] - \bar{u}^N$$

$$H[\bar{u}^N + h](z) = \textstyle\sum_{n=0}^{N} \bar{a}_n z^n + \sum_{n=N+1}^{\infty} T^\infty(\alpha)_n z^n. \qquad \Longrightarrow \qquad \displaystyle \hat{H}[h](z)\Big|_{\mathbb{D}_0(R)} = \sum_{n=N+1}^{\infty} T^\infty(\alpha)_n z^n$$

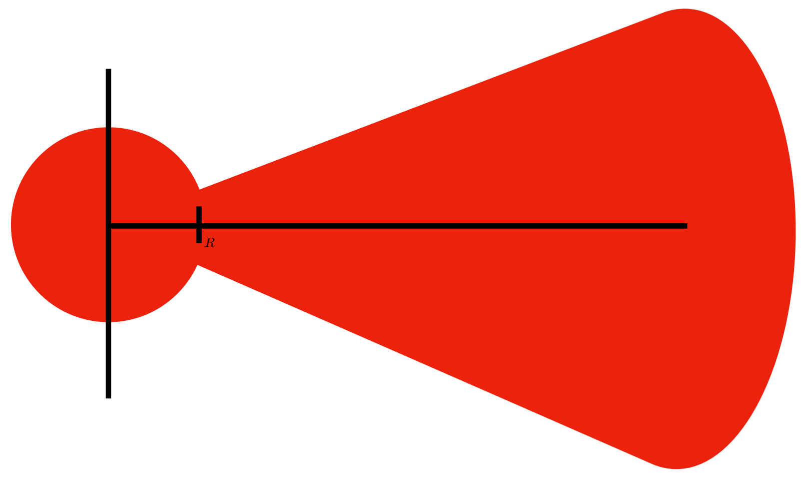
$$\begin{aligned}
\|\hat{H}[0]\|_{\mathbb{D}_0(R),\nu} &= \sup_{z \in \mathbb{D}_0(R)} \left| \sum_{n=N+1}^{\infty} T^\infty(0)_n z^n \right| (1 + \nu^2 |z|) e^{-\nu |z|} \\
&\leq \sup_{z \in \mathbb{D}_0(R)} \sum_{n=N+1}^{\infty} |T^\infty(0)_n| |z|^n \\
&\leq \sum_{n=N+1}^{\infty} |T^\infty(0)_n| R^n \\
&= Y_0
\end{aligned}$$

Note: Y_0 is the same as the Y from the previous proof

A Fixed Point Problem

$$X := \left\{ h \in \mathcal{A}(\Omega) \mid h^{(n)}(0) = 0,\ n = 0, \ldots, N, \|h\|_{\max} < \infty \right\}$$

$$\|h\|_{\ell^1_R} = \sum_{n=N+1}^{\infty} \left| \frac{h^{(n)}(0)}{n!} \right| R^n, \qquad \|h\|_\nu = \sup_{z\in\Omega} |h(z)|(1+\nu^2|z|^2)e^{-\nu|z|}$$



Continuation onto the real axis:

$$\hat{H}[h] = H[\bar{u}^N + h] - \bar{u}^N$$

$$H[\bar{u}^N + h](z) = \textstyle\sum_{n=0}^N \bar{a}_n z^n + \sum_{n=N+1}^{\infty} T^\infty(\alpha)_n z^n. \quad \Longrightarrow \quad \hat{H}[h](z)\Big|_{\mathbb{D}_0(R)} = \sum_{n=N+1}^{\infty} T^\infty(\alpha)_n z^n$$

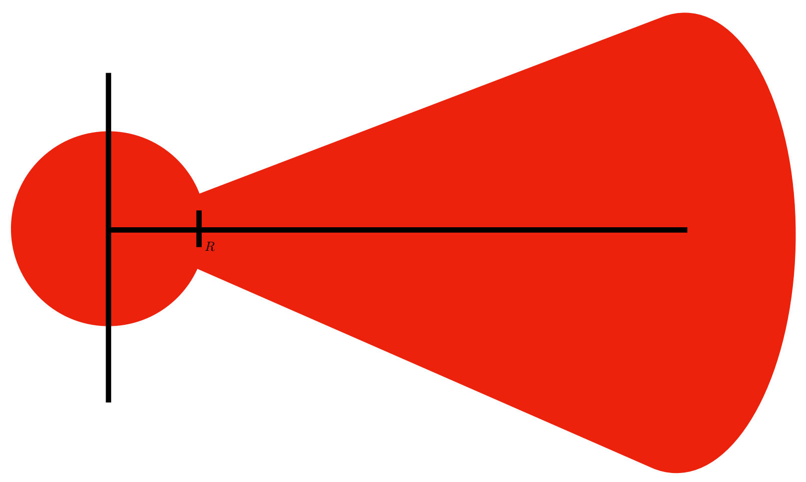
$$\begin{aligned}
\|\hat{H}[0]\|_{\Omega\setminus\mathbb{D}_0(R),\nu} &= \sup_{z\in\Omega\setminus\mathbb{D}_0(R)} \left| \frac{1}{1+z}(1 - \varepsilon(\bar{u}^N * \bar{u}^N)(z) - (1+z)\bar{u}^N(z)) \right| (1+\nu^2|z|^2)e^{-\nu|z|} \\
&\le \sup_{z\in\Omega\setminus\mathbb{D}_0(R)} \left| \frac{1}{1+z} \right| \sup_{z\in\Omega\setminus\mathbb{D}_0(R)} |1 - \varepsilon(\bar{u}^N * \bar{u}^N)(z) - (1+z)\bar{u}^N(z))|(1+\nu^2|z|^2)e^{-\nu|z|} \\
&\le \frac{1}{\sqrt{1+R^2}} \|1 - \varepsilon(\bar{u}^N * \bar{u}^N)(z) - (1+z)\bar{u}^N(z)\|_{\Omega\setminus\mathbb{D}_0(R),\nu} \\
&= Y_1
\end{aligned}$$

A Fixed Point Problem

Continuation onto the real axis:

$$X := \left\{ h \in \mathcal{A}(\Omega) \mid h^{(n)}(0) = 0,\ n = 0, \dots, N, \|h\|_{\max} < \infty \right\}$$

$$\|h\|_{\ell^1_R} = \sum_{n=N+1}^{\infty} \left| \frac{h^{(n)}(0)}{n!} \right| R^n, \qquad \|h\|_{\nu} = \sup_{z \in \Omega} |h(z)|(1+\nu^2|z|^2)e^{-\nu|z|}$$



Ω

$$\hat{H}[h] = H[\bar{u}^N + h] - \bar{u}^N$$

$$H[\bar{u}^N + h](z) = \textstyle\sum_{n=0}^{N} \bar{a}_n z^n + \sum_{n=N+1}^{\infty} T^{\infty}(\alpha)_n z^n. \implies \hat{H}[h](z)\Big|_{\mathbb{D}_0(R)} = \displaystyle\sum_{n=N+1}^{\infty} T^{\infty}(\alpha)_n z^n$$

For the derivative we do the same approach of split and reuse

$$\|D\hat{H}[h]\|_{\mathbb{D}_0(R),\nu} \leq Z_0(\|\alpha\|_{1,R}) = Z_0(\|h\|_{\ell^1_R})$$

$$\begin{aligned}
\|D\hat{H}[h]\|_{\Omega\setminus\mathbb{D}_0(R),\nu} &= \sup_{\|g\|_\nu = 1} \|D\hat{H}[h]g\|_{\Omega\setminus\mathbb{D}_0(R),\nu} \\
&= \sup_{\|g\|_\nu = 1} \left\| \frac{-2\varepsilon}{1+z}((\bar{u}^N + h) * g) \right\|_{\Omega\setminus\mathbb{D}_0(R),\nu} \\
&\leq \sup_{\|g\|_\nu = 1} \frac{2\varepsilon}{\sqrt{1+R^2}} \cdot \frac{4\pi}{\nu} \|\bar{u}^N + h\|_\nu \|g\|_\nu \\
&\leq \frac{8\pi\varepsilon}{\nu\sqrt{1+R^2}} (\|\bar{u}^N\|_\nu + \|h\|_\nu) \\
&= Z_1(\|h\|_\nu)
\end{aligned}$$

Then,

$$\|D\hat{H}(h)\|_{\max} \leq Z_0(\|h\|_{\max}) + Z_1(\|h\|_{\max}) = \hat{Z}(\|h\|_{\max})$$

So, there exists $\hat{h} \in X$ such that $\hat{H}[\bar{h}] = \bar{h}$ and $\|\bar{h}\|_{\max} \boxed{\leq r_1}$

$$\hat{Y} + \textstyle\int_0^{r_1} \hat{Z}(s)\, ds \leq r_1 \qquad \hat{Y} = \max\{Y_0, Y_1\}$$

Error Bound

Going back to our objective

$$|\bar{y}(t) - y^N(t)| = |\mathcal{L}[\bar{u}] - \mathcal{L}[u^N]| \le \left|\int_0^R (\bar{u}_0(z) - u^N(z))\, e^{-tz}\, dz\right| + \left|\int_R^\infty (\bar{u}(z) - u^N(z))\, e^{-tz}\, dz\right| \le \quad ?$$

$$\int_0^R |\bar{u}_0(z) - u^N(z)|\, e^{-tz}\, dz \le \int_0^R r_0 e^{-tz}\, dz = \frac{r_0}{t}(1 - e^{-Rt})$$

$$\begin{aligned}\int_R^\infty |\bar{u}(z) - u^N(z)|\, e^{-tz}\, dz &\le \int_R^\infty \frac{\|\bar{u}(z) - u^N(z)\|_\nu}{1 + \nu^2|z|^2} e^{\nu|z|}\, e^{-tz}\, dz \\ &\le \int_R^\infty \left(\frac{r_1}{1 + \nu^2 R^2}\right) e^{-(t-\nu)z}\, dz \\ &\le \frac{1}{t - \nu}\left(\frac{r_1}{1 + \nu^2 R^2}\right) e^{-(t-\nu)R}\end{aligned}$$

Why not $\int_0^\infty$?

$$\begin{aligned}\int_0^\infty |\bar{u}(z) - u^N(z)| e^{-tz}\, dz &\le \int_0^\infty \|\bar{u} - u^N\|_\nu \frac{e^{\nu z}}{1 + \nu^2 z^2}\, e^{-tz} dz \\ &= \|\bar{u} - u^N\|_\nu \int_0^\infty \frac{e^{-(t-\nu)z}}{1 + \nu^2 z^2}\, dz \\ &\le r_1 \min\left\{\int_0^\infty \frac{1}{1 + \nu^2 z^2}\, dz, \int_0^\infty e^{-(t-\nu)z}\, dz\right\} \\ &= r_1 \min\{\pi/(2\nu), 1/(t-\nu)\}\end{aligned}$$

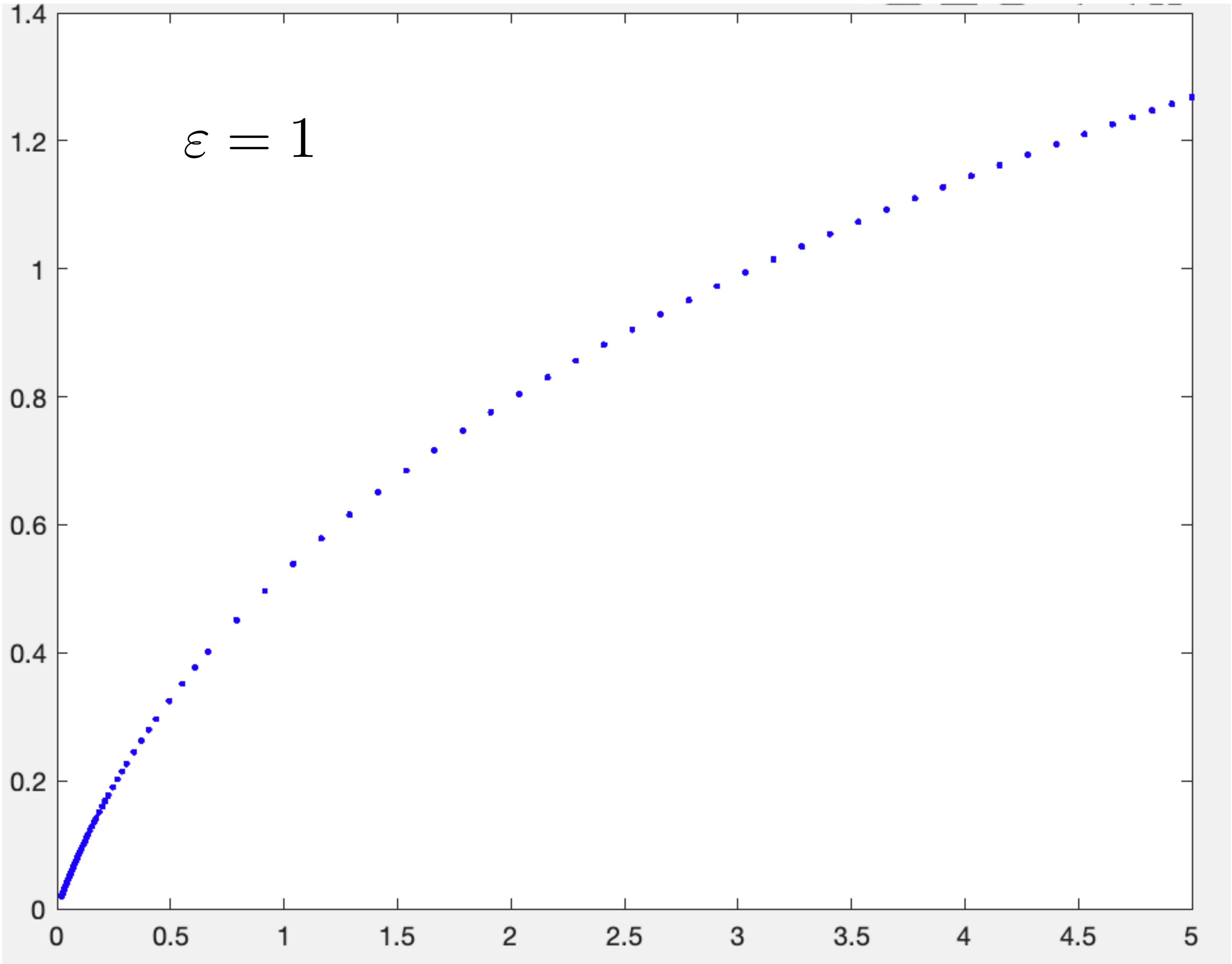
Putting it all together

$$|\bar{y}(t) - y^N(t)| \le \frac{r_0}{t}(1 - e^{-Rt}) + \frac{1}{t - \nu}\left(\frac{r_1}{1 + \nu^2 R^2}\right) e^{-(t-\nu)R}$$

Some Computations

$N = 35, x = 0.02, R = 0.2, \nu = 40$

ε	r_0	r_1	$\lvert y(0.02) - y^{35}(0.02)\rvert \leq$
0.5	$1.821691933910861 \times 10^{-34}$	$5.982878018285794 \times 10^{-12}$	1.3643×10^{-16}
0.75	$2.182511302790229 \times 10^{-23}$	$5.402480219404250 \times 10^{-12}$	1.1248×10^{-15}
1	$1.105694178445770 \times 10^{-22}$	$5.332936946629159 \times 10^{-11}$	1.1104×10^{-14}



Note: For this example we could do a fully analytical proof by taking advantage of the particularities of the operator, i.e $1/1+z$, and using this as the center of our ball in function space.

$N = 35, x = 0.02, R = 0.2, \nu = 49$

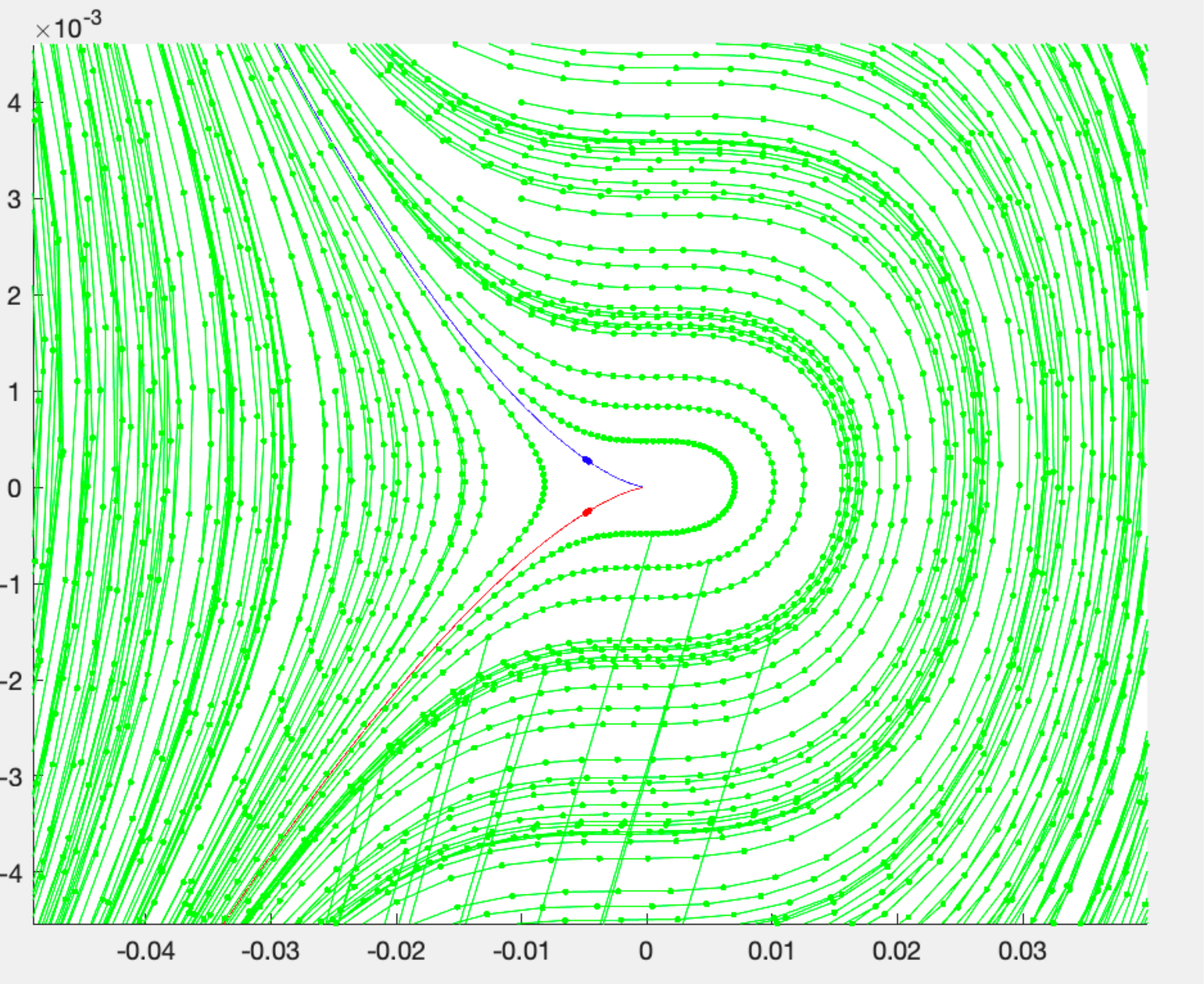
ε	δ	$\lvert y(0.02) - y^{35}(0.02)\rvert \leq$
10^{-14}	5.0089×10^{-15}	1.7363×10^{-16}
10^{-7}	5.0089×10^{-8}	1.7362×10^{-9}
0.1	5.4557×10^{-2}	1.8802×10^{-3}

Cubic Euler $x^2y' + y + \varepsilon y^3 = x \quad \rightarrow \quad zu(z) + u(z) + \varepsilon(u * u * u)(z) = 1$

$N = 35, x = 0.01, R = 0.4, \nu = 75$

ε	r_0	r_1	$\lvert y(0.01) - y^{35}(0.01)\rvert \leq$
0.5	$5.525079387942491 \times 10^{-16}$	$8.320748559069655 \times 10^{-12}$	5.5419×10^{-18}
0.75	$3.483447486489139 \times 10^{-16}$	$1.283977121916329 \times 10^{-11}$	3.5093×10^{-18}
1	$1.091933120924504 \times 10^{-17}$	$1.775231538381258 \times 10^{-11}$	1.4497×10^{-19}

Hénon Map



$$F(x,y) = \begin{pmatrix} x + y - x^2 \\ y - x^2 \end{pmatrix}$$

$$DF(x,y) = \begin{pmatrix} 1-2x & 1 \\ -2x & 1 \end{pmatrix}$$

$$DF(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has double eigenvalue of 1.

$$u(t+1) - 2u(t) + u(t-1) = -u^2(t)$$

$$|\bar{u}(35) - u^{27}(35)| \leq 8.456 \times 10^{-16}$$

We are looking for the curves parametrized by

$$P(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

which satisfy the invariance equation

$$F(u(t), v(t)) = \begin{pmatrix} u(t+1) \\ v(t+1) \end{pmatrix},$$

such that

$$\lim_{t \to \pm\infty} P(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thank you for listening!!

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