Root certification using disk arithmetic

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The Newton-Raphson iterative method

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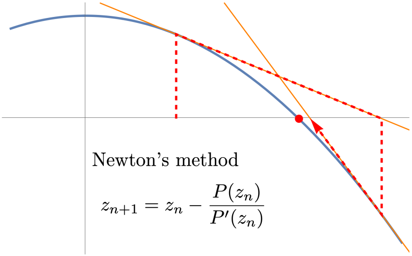

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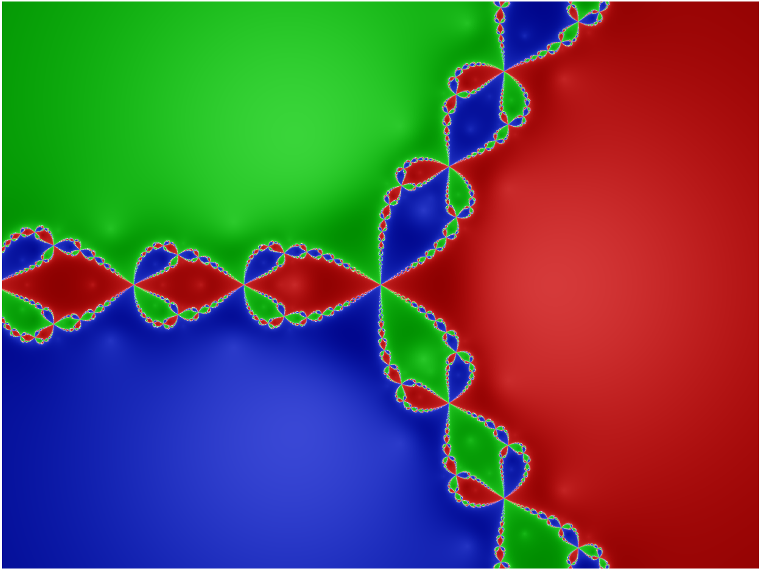
Observe that:

- if $f(z) = 0$ then $N_f(z) = z$
- if z is a simple root of f, $N_f'(z) = 0$, as $f'(z) \neq 0$ and

$$N_f'(z) = \frac{f(z)f''(z)}{f'(z)^2}$$

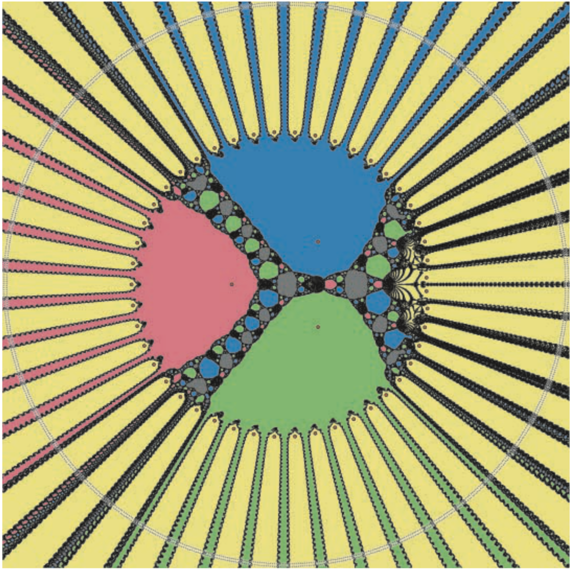
- if z is a root of order $p \geq 2$ of f, then $N_f'(z) = \frac{p-1}{p} < 1$

Interesting dynamics for polynomials of degree ≥ 3



The basins of attraction of the Newton method of $z^3 - 1$

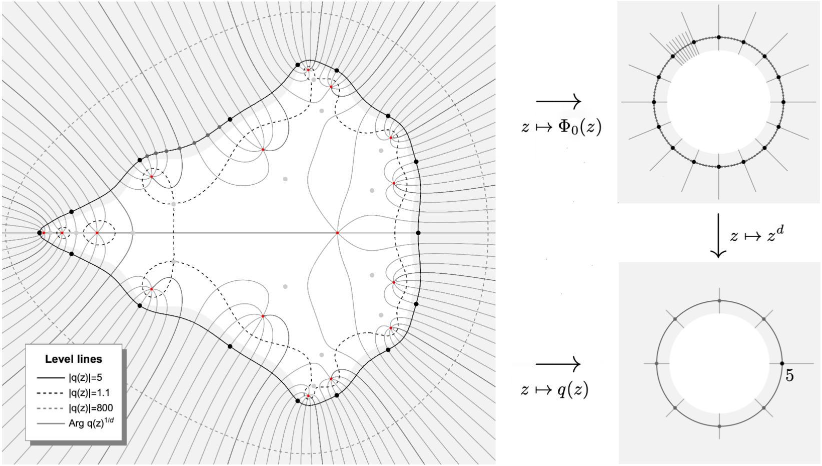
Algorithm by Hubbard, Schleicher and Sutherland (2001)



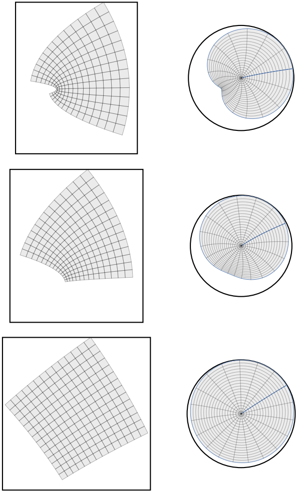
The Newton method of a polynomial of degree 50

Algorithm by M.-Vigneron (2026)

Let $q \in \mathbb{C}[z]$ be a monic polynomial of degree d and $\Phi_0 = q^{1/d}$ near ∞.



Disk arithmetic



Disk arithmetic is superior to interval arithmetic !

$$f(z) = z^2 \qquad z_0 = e^{3i\pi/16}$$

Images of

$$z_0 + [\pm r \pm ir]$$
$$z_0 + D(0, r)$$

for $r = 0.8$ (top), 0.5 (middle), 0.1 (bottom).

Loss after n iterates from $|z| = 1$:

$$re^{0.2365n} \quad \text{(box)}$$

$$r(1+r/2)^n \simeq r\left(1 + \frac{nr}{2}\right) \quad \text{(disk)}$$

Algorithm 1: Certification Algorithm using disk arithmetic

Data: A root candidate $z_0 \in \mathcal{R}_N + i\mathcal{R}_N$, $\varepsilon_R > 0$ and $\varepsilon_N > 3\varepsilon_R$

`Compute` $f(z_0) \in D(z_1, \alpha_1)$, $f'(D(z_0, \varepsilon_R)) \subseteq D(z_2, \alpha_2)$

if $R_N^-(\varepsilon_R(|z_2| - \alpha_2)) > R_N^+(|z_1| + \alpha_1)$ **then**
  `print:` f has a unique root z_* in $D(z_0, \varepsilon_R)$

else
  `print:` cannot verify the unicity of the root of f in $D(z_0, \varepsilon_R)$
  `return FALSE`

`Compute` $f'(D(z_0, \varepsilon_N)) \subseteq D(z_3, \alpha_3)$

if $R_N^-(|z_3|) \geq 5\alpha_3$ **then**
  `print:` N_f iterates converge to z_* on $D(z_0, \varepsilon_N)$
  `return TRUE`

else
  `print:` cannot verify the convergence of N_f on $D(z_0, \varepsilon_N)$
  `return FALSE`

A couple of basic theoretical results

Theorem

Given $f \in \mathrm{Hol}(U)$ and $D(z_0, r) \subset U$ and $f'(D(z_0, r)) \subset B'$ a disk, if

$$r\,\mathrm{dist}(0, B') > |f(z_0)|,$$

then there exists a unique point $z_* \in D(z_0, r)$ such that $f(z_*) = 0$.

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For $\gamma(t) = z_0 + re^{2i\pi t}$ that parametrizes ∂B, one has

$$f(\gamma(t)) = f(z_0) + re^{2i\pi t} \int_0^1 f'(z_0 + sre^{2i\pi t})ds \in f(z_0) + re^{2i\pi t} \cdot B'$$

The arc $f \circ \gamma$ is homotopic in $\mathbb{C}\backslash\{0, f(z_0)\}$ to the circle $C(f(z_0), r|f'(z_0)|)$

$$\# f^{-1}(0) \cap D(z_0, r) = \frac{1}{2i\pi} \int_\gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2i\pi} \int_{f\circ\gamma} \frac{dz}{z} = 1$$

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Given $f \in \mathrm{Hol}(U)$ *and* $D(z_0, r) \subset U$ *and* $f'(D(z_0, r)) \subset B'$ *a disk, if*

$$r\,\mathrm{dist}(0, B') > |f(z_0)|,$$

then there exists a unique point $z_* \in D(z_0, r)$ *such that* $f(z_*) = 0$*.*

Theorem

If, moreover, there are $3r \leq R \leq \mathrm{dist}(z_0, \partial U)$ *and*

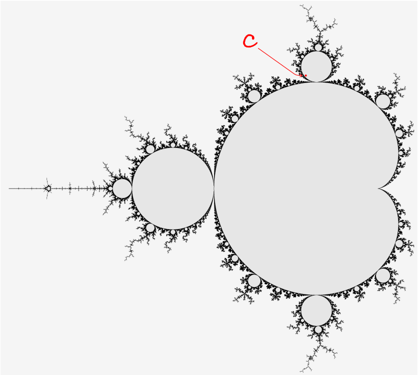
$$f'(D(z_0, R)) \subset B'' \qquad \text{with} \qquad \mathrm{dist}(B'', 0) \geq 2\,\mathrm{diam}(B''),$$

then $D(z_0, R)$ *is contained in the attraction basin of* z_* *for Newton's method* N_f*.*

The Mandelbrot set

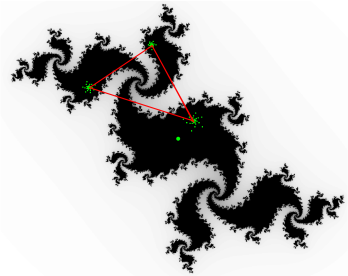
$$\boxed{f_c(z) = z^2 + c}$$

$$\mathcal{M} = \left\{ c \in \mathbb{C} \mid (\underbrace{f_c \circ \cdots \circ f_c}_{n \text{ iterates}}(0))_{n\in\mathbb{N}} \text{ is bounded} \right\}$$



Parmeter space

$$p_1(c) = c$$
$$p_{n+1}(c) = p_n^2(c) + c$$
$$\deg p_n = 2^{n-1}$$



$$0 \overset{f_c}{\rightsquigarrow} p_1(c) \overset{f_c}{\rightsquigarrow} p_2(c) \overset{f_c}{\rightsquigarrow} \cdots \overset{f_c}{\rightsquigarrow} p_k(c) \overset{f_c}{\rightsquigarrow} \cdots$$

Dynamics