A computer-assisted proof of dynamo growth in the Stretch-Fold-Shear map

Ben Mestel
joint work with Farhana Pramy and Andrew Gilbert

ICMS, **July 26**

The Open University

Stretch — Fold — Shear operator S

- $F = \{c(x)$ complex-valued function of a real variable $x \in [-1,1]\}$
- $S: F \to F$
- $Sc(x) = e^{\frac{i\alpha(x-1)}{2}} c\left(\frac{x-1}{2}\right) - e^{\frac{i\alpha(1-x)}{2}} c\left(\frac{1-x}{2}\right)$
- $\alpha \geq 0$ real parameter
- Model of magnetic field growth in kinematic dynamo theory studied by Bayly, Childress and Gilbert
- Eigenvalue λ of modulus greater than one means growing magnetic field leading to dynamo

Kinematic dynamo theory

- Induction equation:

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \epsilon \nabla^2 \boldsymbol{B}$$

where $\boldsymbol{B}$ magnetic field, $\boldsymbol{u}$ fluid flow, ϵ^{-1} is the magnetic Reynolds number

- $\boldsymbol{u} = \boldsymbol{u}(x, y, t) = (u_x, u_y, u_z)$ (planar flow case)
- Magnetic field lines get stretched and folded through the action of u_x, u_y
- Insufficient for dynamo because it is prone to diffusive cancellation of the field
- Shear in the z direction can align field so growth can occur

Stretch-Fold-Shear mechanism

A.D. Gilbert / Physica D 166 (2002) 167–196 169

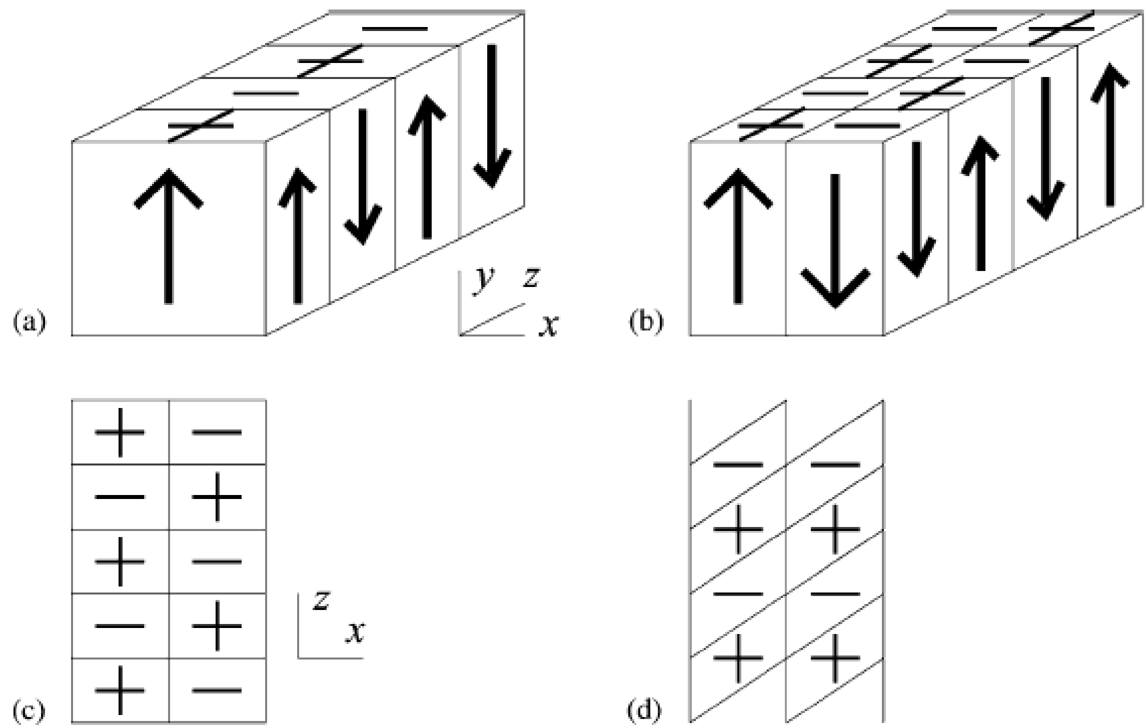


Fig. 1. The SFS map. (a) Magnetic field depending on z is stretched and folded with a baker's map in the (x, y)-plane to give (b). In (c) the field orientation is shown in the (x, z)-plane, which after the shear operation gives (d). The effect of the SFS operations from (a) to (d) is to double the magnitude of field vectors and partially bring like-signed field together.

Stretch – Fold – Shear model

- $M: [-1,1]^2 \times \mathbb{R} \to [-1,1]^2 \times \mathbb{R}, M = M_{sh}\, M_{SF}$
- $M_{SF}(x,y,z) = \begin{cases} \left(\frac{1}{2}(x-1), 1+2y, z\right), & y < 0 \\ \left(\frac{1}{2}(1-x), 1-2y, z\right), & y \geq 0 \end{cases}$

(Folded Baker's map)

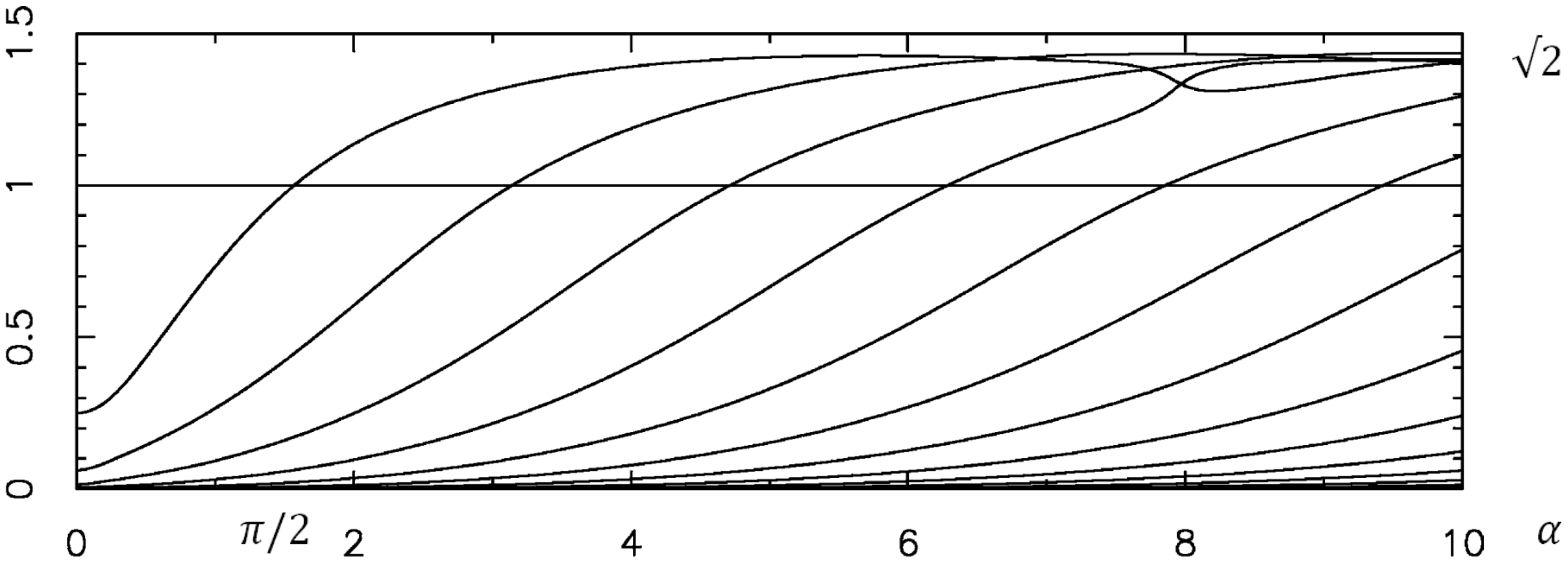
- $M_{sh}\,(x,y,z) = (x,y,z+\alpha x)$
- Shear in z direction, $\alpha \geq 0$ real parameter
- Taking $\boldsymbol{B}(x,y,x) = e^{ix} b(x)\hat{\boldsymbol{y}} + c.c.$ where $b(x)$ is a complex-valued function of $x \in [-1,1]$
- The map M induces an operator T on $b(x)$ which in turn induces the operator $S = T^*$ on $c(x)$, the adjoint of $b(x)$.

Stretch — Fold — Shear operator S

- $F = \{c(x)$ complex-valued function of a real variable $x \in [-1,1]\}$
- $S: F \to F$
- $Sc(x) = e^{\frac{i\alpha(x-1)}{2}} c\left(\frac{x-1}{2}\right) - e^{\frac{i\alpha(1-x)}{2}} c\left(\frac{1-x}{2}\right)$
- $\alpha \geq 0$ real parameter
- Model of magnetic field growth in kinematic dynamo theory studied by Bayly, Childress and Gilbert
- Eigenvalue λ of modulus greater than one means growing magnetic field leading to dynamo

Spectrum of S

Graph of modulus of eigenvalues for $\alpha \geq 0$



Graph from Gilbert 2002

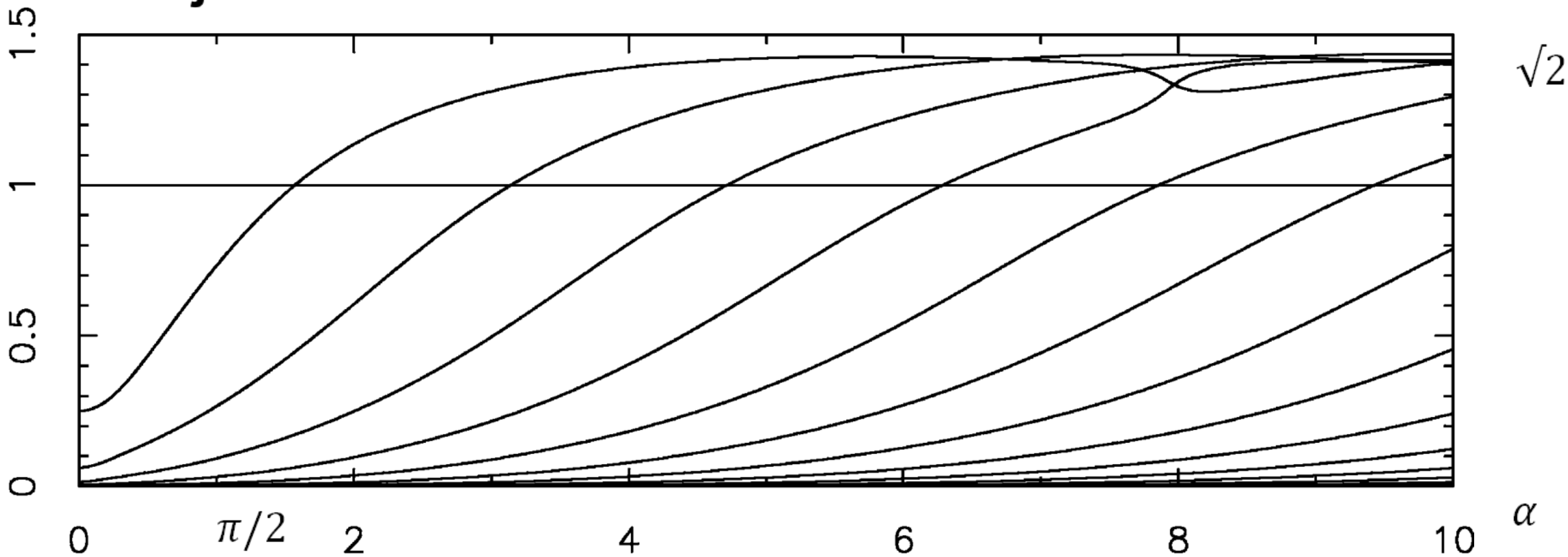
Properties of S, for $\alpha \geq 0$

- For $\alpha = 0$, the spectrum of S can be determined exactly:
- $\lambda_n = 1/4^n$, $n = 0,1,2, \ldots$ with eigenfunctions $c_n(\mathrm{x})$ a polynomial of degree $2n + 1$
- For $\alpha > 0$, no known exact solutions except the Bayly solution:
- $c(x) = e^{i\alpha(x-1)} - e^{i\alpha(1-x)} = 2\, i \sin\alpha(x-1)$, $\lambda = e^{-i\alpha}$
- Other rigorous results only via computer-assisted proof for $\alpha > 0$

Gilbert’s Conjectures

- For $\alpha > \pi/2$, S has an eigenvalue $|\lambda|>1$ (and so SFS produces a dynamo)
- For each eigenfunction family $c(x;\alpha)$ parametrized by α, the limit $\lim_{\alpha\to\infty} |\lambda| = \sqrt{2}$

In this work we make progress on the first of these conjectures.



Main result

Theorem 1 (Computer assisted)

Let $\alpha \in [0,5]$*. Then there exists an eigenvalue-eigenfunction pair* $\big(\lambda, c(x)\big)$ *of the operator* S *satisfying the following:*

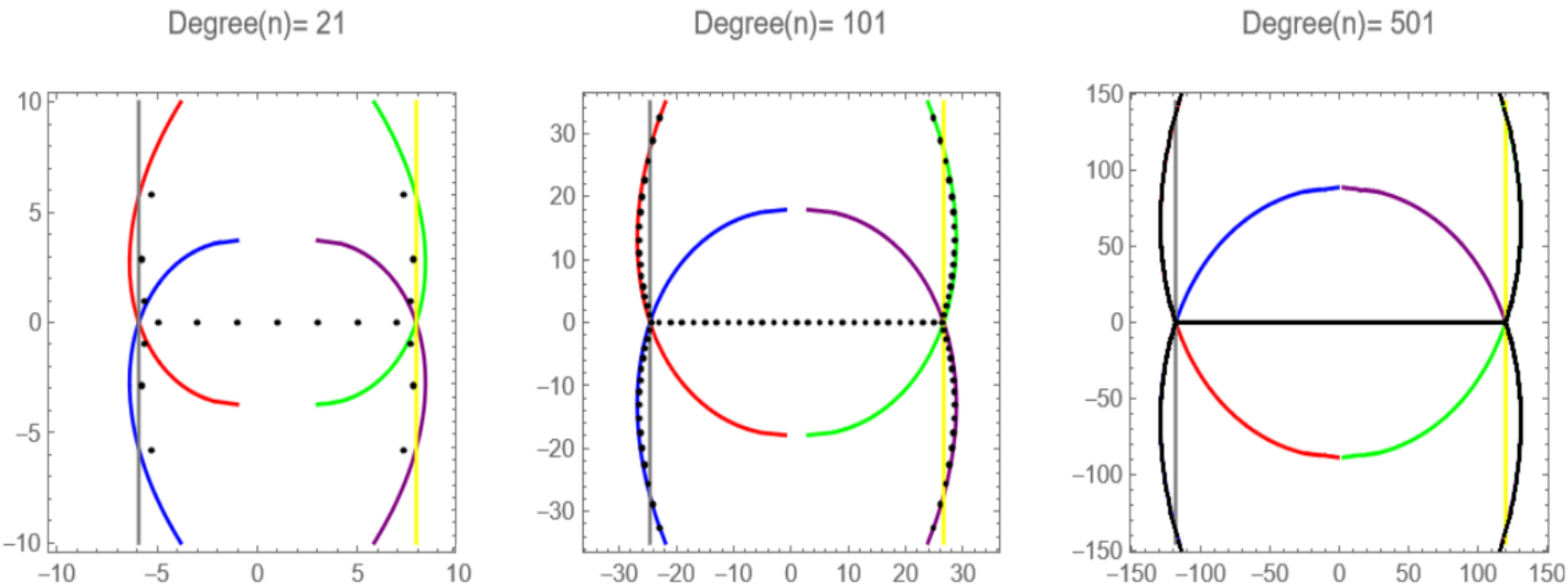
- $c(x)$ *is analytic on* D_r *(disc about* 0 *of radius* r*), for some* $r > 1$.
- *For* $\alpha \in [0, \pi/2)$, $|\lambda| < 1$.
- *For* $\alpha \in (\pi/2, 5]$, $|\lambda| > 1$.
- *For* $\alpha = \pi/2$, $\lambda = -i$.

Proof based on two separate but related results

The case $\alpha = 0$

- $Sc(x) = c\left(\frac{x-1}{2}\right) - c\left(\frac{1-x}{2}\right)$
- Eigenfunctions can be determined in terms of Bernoulli polynomials via generating function
- $\sum_{n=0}^{\infty} c_n(x)\frac{t^n}{n!} = G(x,t)$ where
- $G(x,t) = \frac{2t\left(e^{(x-1)t} - e^{(1-x)t}\right)}{e^{2t} - e^{-2t}}$
- $SG(x,t) = 2G\left(x, \frac{t}{2}\right)$ functional equation
- $c_n(x) = \begin{cases} 2^{2n} B_n\left(\frac{x+1}{4}\right) & (n \text{ odd}) \\ 0 & (n \text{ even}) \end{cases}$
- $\lambda_n = 1/2^{n-1}$ (n odd), 0 (n even)

Zeros of $c_n(x)$ (after Mangual ‘07)



$$\mathrm{Re}(x) = -\sqrt{\frac{4n^2 e^{-\frac{1}{n}\pi\,\mathrm{Im}(x)-2}}{\pi^2} - \mathrm{Im}(x)^2} - 1$$

$$\mathrm{Re}(x) = \sqrt{\frac{4n^2 e^{-\frac{1}{n}\pi\,\mathrm{Im}(x)-2}}{\pi^2} - \mathrm{Im}(x)^2} + 3$$

$$-\mathrm{Re}(x) - 1 = \frac{2n}{e\pi}$$

$$\mathrm{Re}(x) = -\sqrt{\frac{4n^2 e^{\frac{1}{n}\pi\,\mathrm{Im}(x)-2}}{\pi^2} - \mathrm{Im}(x)^2} - 1$$

$$\mathrm{Re}(x) = \sqrt{\frac{4n^2 e^{\frac{1}{n}\pi\,\mathrm{Im}(x)-2}}{\pi^2} - \mathrm{Im}(x)^2} + 3$$

$$\mathrm{Re}(x) - 3 = \frac{2n}{e\pi}$$

(From thesis of Farhana Pramy 2023)

Results away from $\alpha = \pi/2$

Theorem 2 (Computer assisted)

Let $\alpha \in [0,5]$*. Then there exists an eigenvalue-eigenfunction pair* $(\lambda, c(x))$ *of the operator* S *satisfying the following:*

- $c(x)$ *is analytic on* D_r *for some* $r > 1$.
- *For* $\alpha \in [0,1.5705]$, $|\lambda| < 1$.
- *For* $\alpha \in [1.571, 5]$, $|\lambda| > 1$.
- *For* $\alpha \in [1.5705,1.571]$, $|\lambda| \in [0.99624, 1.00374]$.

Description of Proof Method

- Define a map $G: \mathbb{C} \times \mathcal{F} \to \mathbb{C} \times \mathcal{F}$ by

$$G(\lambda, c) = (c(0) - 1, (S - \lambda)c)$$

- Normalise eigenfunctions via $c(0) = 1$
- A normalised eigenvalue-eigenfunction pair (λ, c) satisfies $G(\lambda, c) = (0,0)$
- For fixed $\alpha \geq 0$, prove existence of eigenvalue-eigenfunction pair via Newton iteration scheme

$$(\lambda, c) \mapsto (\lambda, c) - dG_{(\lambda,c)}^{-1} G(\lambda, c)$$

Description of Proof method II

- Start from $\alpha = \alpha_0 = 0$ and choose an initial eigenvalue-eigenfunction pair (i.e. $(\frac{1}{4n}, c_n(x))$for some n)
- Working on a sequence of small intervals $[\alpha_{i-1}, \alpha_i]$, $i \geq 1$, find an approximate eigenfunction-eigenvalue pair $(\bar{\lambda}_i, \bar{c}_i(x))$
- Use rigorous numerical functional analysis to show that for all $\alpha \in [\alpha_{i-1}, \alpha_i]$ there exists an eigenfunction-eigenvalue pair $\big(\lambda, c(x)\big)$ in a ball around $(\bar{\lambda}_i, \bar{c}_i(x))$
- Repeat with the next interval
- Rigorous numerical functional analysis implemented in Julia

Results around $\alpha = \pi/2$

Theorem 3 (computer assisted) *Let* $\alpha \in [1.5705, 1.571]$. *Then there exists an eigenvalue-eigenfunction pair* $(\lambda, \hat{c}(x))$ *of* S *such that:*

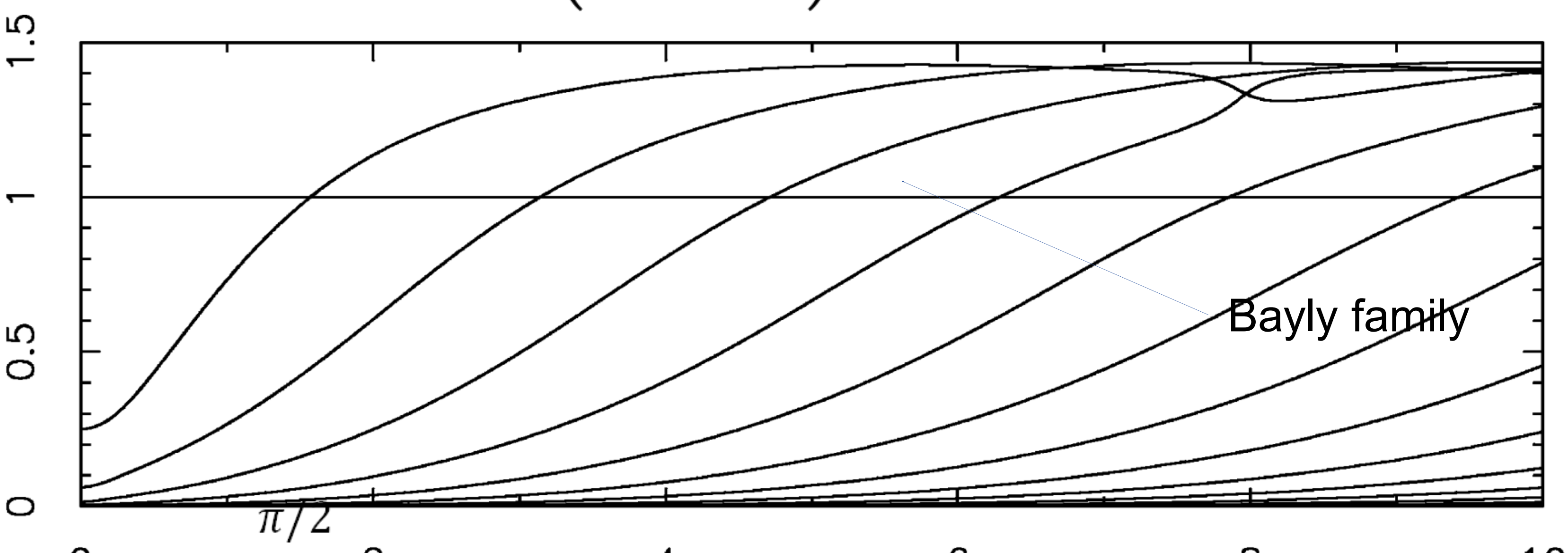
- $\lambda \in [-0.00375, 0.00375] + i[-1.00375, -0.99625]$
- $|\lambda| \in [0.99624, 1.00374]$
    - *If* $\alpha \in [1.5705, \frac{\pi}{2}]$, $|\lambda| < 1$
    - *If* $\alpha \in [\frac{\pi}{2}, 1.571]$, $|\lambda| > 1$
    - *If* $\alpha = \frac{\pi}{2}, \lambda = -i$

The eigenfunction in the case $\alpha = \frac{\pi}{2}$ is the Bayly example

The operator $\hat{S}$

Around $\alpha = \frac{\pi}{2}$, to avoid the Bayly family, write $c(x) = (x-1)(x+1)\hat{c}(x)$. Then $Sc(x) = (x-1)(x+1)\hat{S}\hat{c}(x)$ where

$$\hat{S}\hat{c}(x) = \frac{x-3}{8\left(\frac{(x-1)}{2}\right)} \left[e^{i\alpha x}\,\hat{c}(x)\right]_{2o}\left(\frac{x-1}{2}\right)$$



Proof method around $\alpha = \pi/2$

1. Find an approximate eigenvalue-eigenfunction pair $(\lambda, \hat{c}(x))$ and show that there is a ball B on which for all $\alpha \in [1.5705, 1.571]$ there is a unique eigenvalue-eigenfunction pair. Note that: $\frac{\pi}{2} \in [1.5705, 1.571]$
2. Show that the Bayly example is in this ball (next slide)
3. From the Implicit function theorem deduce that there is a unique smooth family of eigenvalue-eigenfunction pairs for $\alpha \in [1.5705, 1.571]$
4. Find an approximation to the pair $\left(\dot{\lambda}, \dot{\hat{c}}(x)\right) = \left(\frac{d\lambda}{d\alpha}, \frac{\partial \hat{c}}{\partial \alpha}(x)\right)$ and and show that there is a ball which contains $\left(\dot{\lambda}, \dot{\hat{c}}(x)\right)$ for all $\alpha \in [1.5705, 1.571]$
5. Show that $\frac{d|\lambda|^2}{d\alpha} > 0$ for all $\alpha \in [1.5705, 1.571]$

Bayly example

Bayly example at $\alpha = \frac{\pi}{2}$,

$$c(x) = -\sin \pi(x-1)/2$$

Using the Euler product for sine:

$$\sin \pi x = \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right)$$

we write

$$c(x) = (x-1)(x+1)\hat{c}(x)$$

where

$$\hat{c}(x) = \frac{\pi(3-x)}{4} \prod_{k=2}^{\infty} \left(1 - \frac{(x-1)^2}{4k^2}\right)$$

Challenges

- Computational complexity increases fast as α increases so current computer-assisted proof soon becomes computationally infeasible. Can this be overcome?
- Difficulties in choosing the values of the parameters (degree of truncation, precision of calculations, size of intervals)
- How can we deal the case when $\alpha \to \infty$?
- Is there a non-computer-assisted method?

Thanks for listening/Questions

Thank You!

Pramy, F. A., Mestel, B. D., & Gilbert, A. D. (2022). A computer-assisted proof of dynamo growth in the stretch-fold-shear map. *Dynamical Systems*, *38*(1), 102–120. https://doi.org/10.1080/14689367.2022.2139224