Computer-assisted proofs for periodic solutions in state-dependent delay equations

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Joint work with

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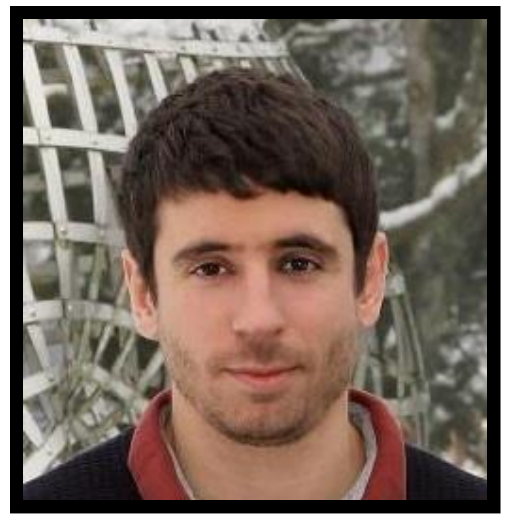
VU Amsterdam



M. Breden
Polytechnique

M. Cadiot
Polytechnique

K. Church
CRM, Montreal

Validated Numerics for Computer-Assisted Proofs
ICMS, Bayes Centre, Edinburgh, 6-10 July 2026

Outline

I. State-dependent delay equations: motivation and challenges

II. Computer-assisted proofs and a Newton–Kantorovich strategy

III. A zero-finding problem for periodic orbits of forced SDDEs

IV. Fourier coefficients of $\bar{u}(t - c_1 - c_2\bar{u}(t))$: the main technical tool

V. Three model problems: forced, autonomous, threshold

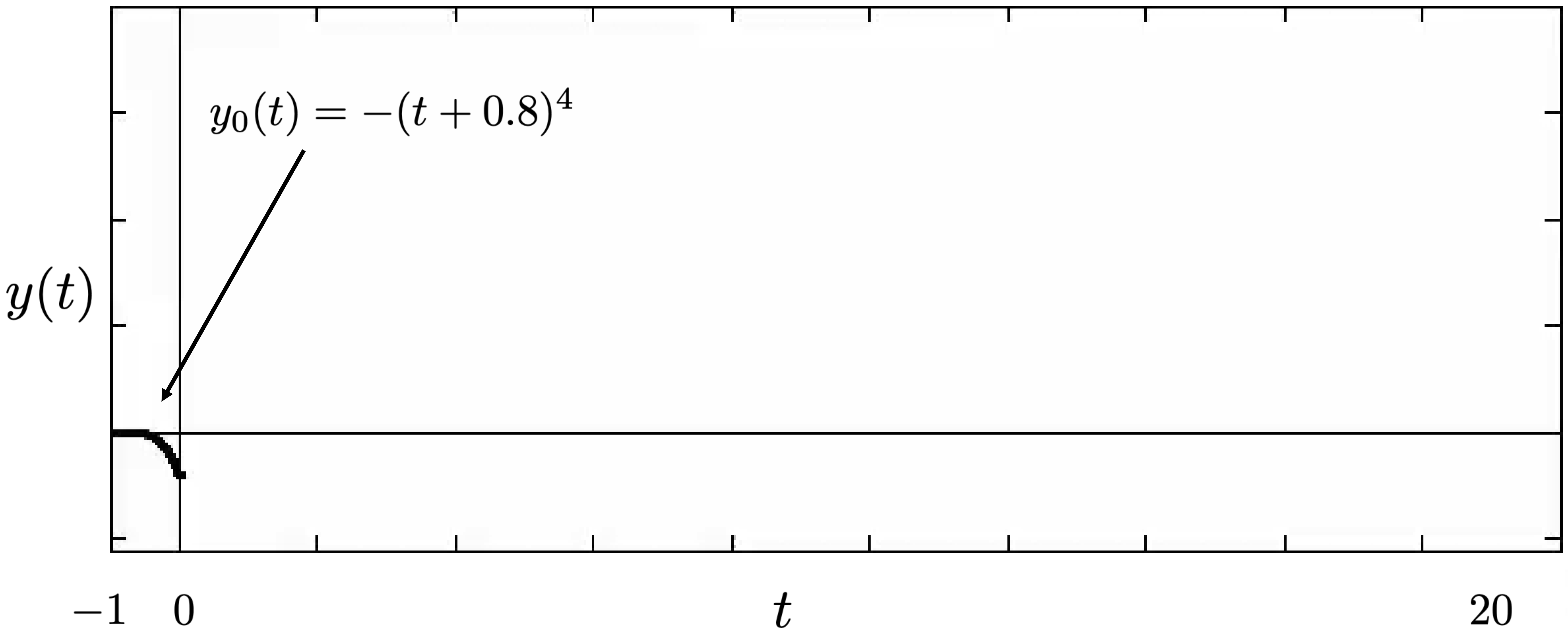
VI. Conclusion and ongoing work

Delay equations as infinite dimensional dynamical systems

Delay differential equations (DDEs), in their simplest form, are expressed as

$$u'(t) = f\big(u(t),\, u(t-\tau)\big)$$

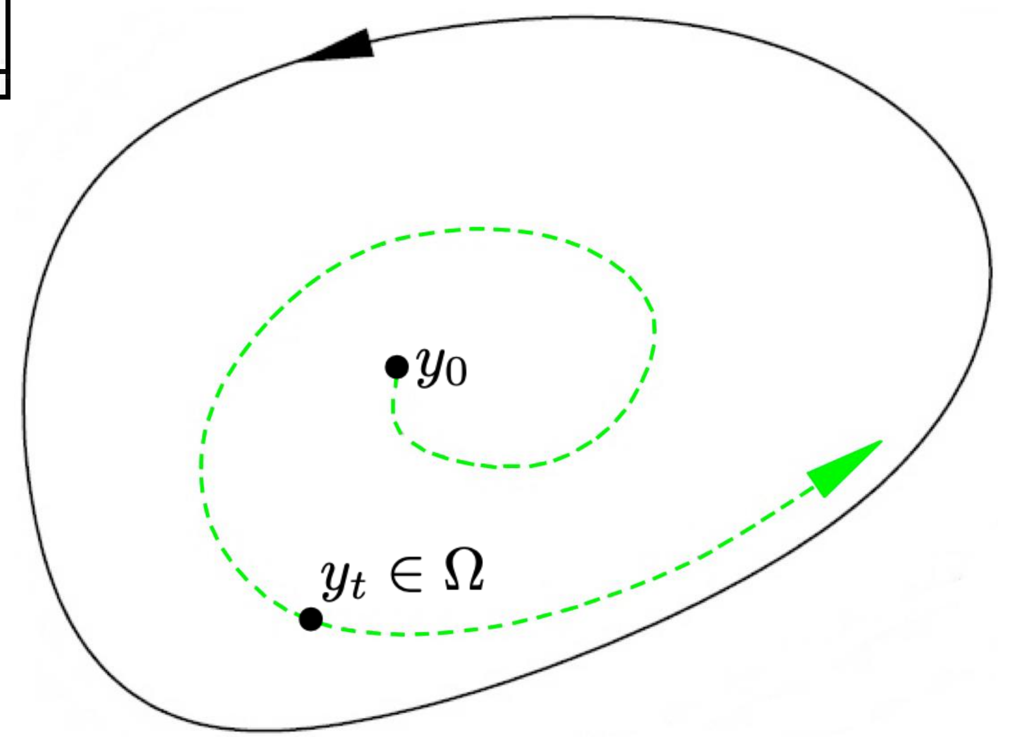
where $\tau > 0$ is the (constant) delay and where $f : U_1 \times U_2 \to \mathbb{R}^n$ is smooth ($U_j \subset \mathbb{R}^n$ open).



ex: $y'(t) = -\frac{12}{5} y(t-1)(1+y(t))$

$\infty-$dimensional dynamical system

Phase space: $C([-1,0])$



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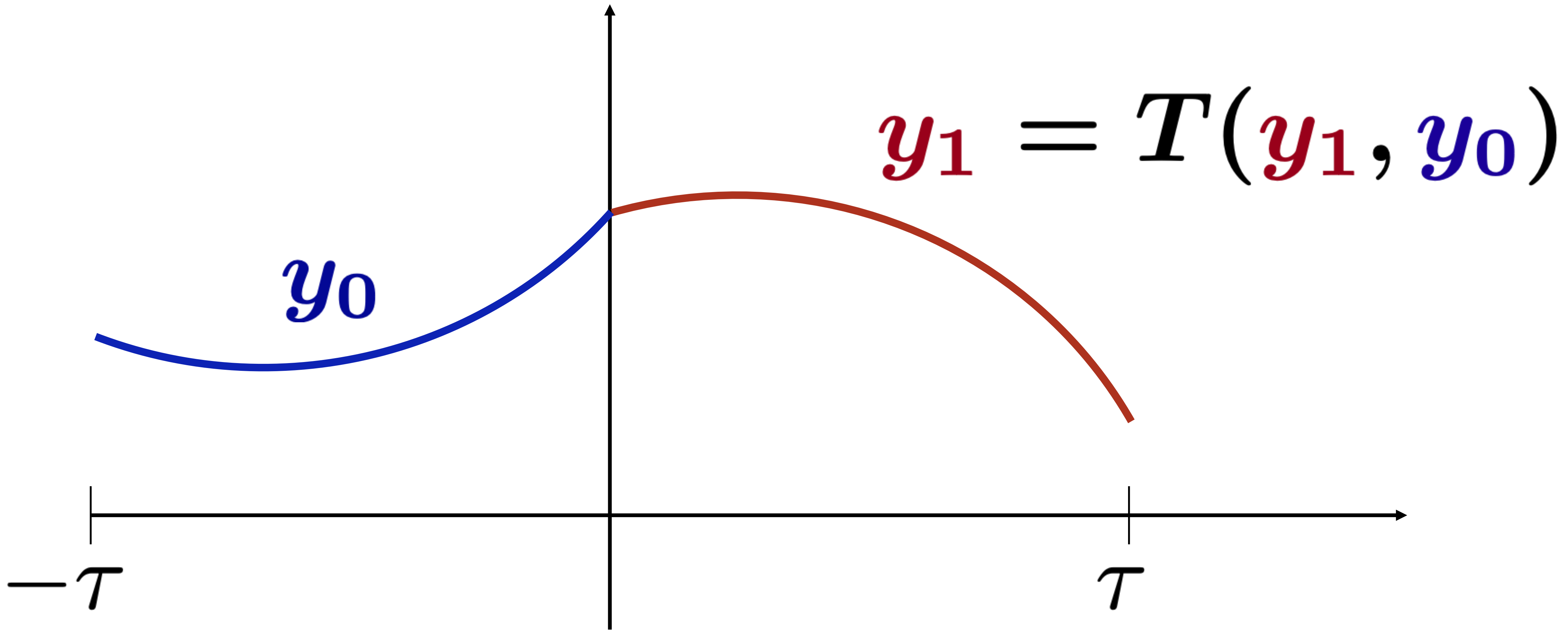
Let $y_0 \in C([-\tau, 0], \mathbb{R}^d)$ an initial condition. The function $y_1 \colon [0,\tau] \to \mathbb{R}^d$ is a *solution* of the DDE on $[0,\tau]$ if it is the *unique fixed point* of the operator

$$T(y_1, y_0)(t) = y_0(0) + \int_0^t f(y_1(s), y_0(s-\tau))\, ds, \qquad t \in [0,\tau].$$



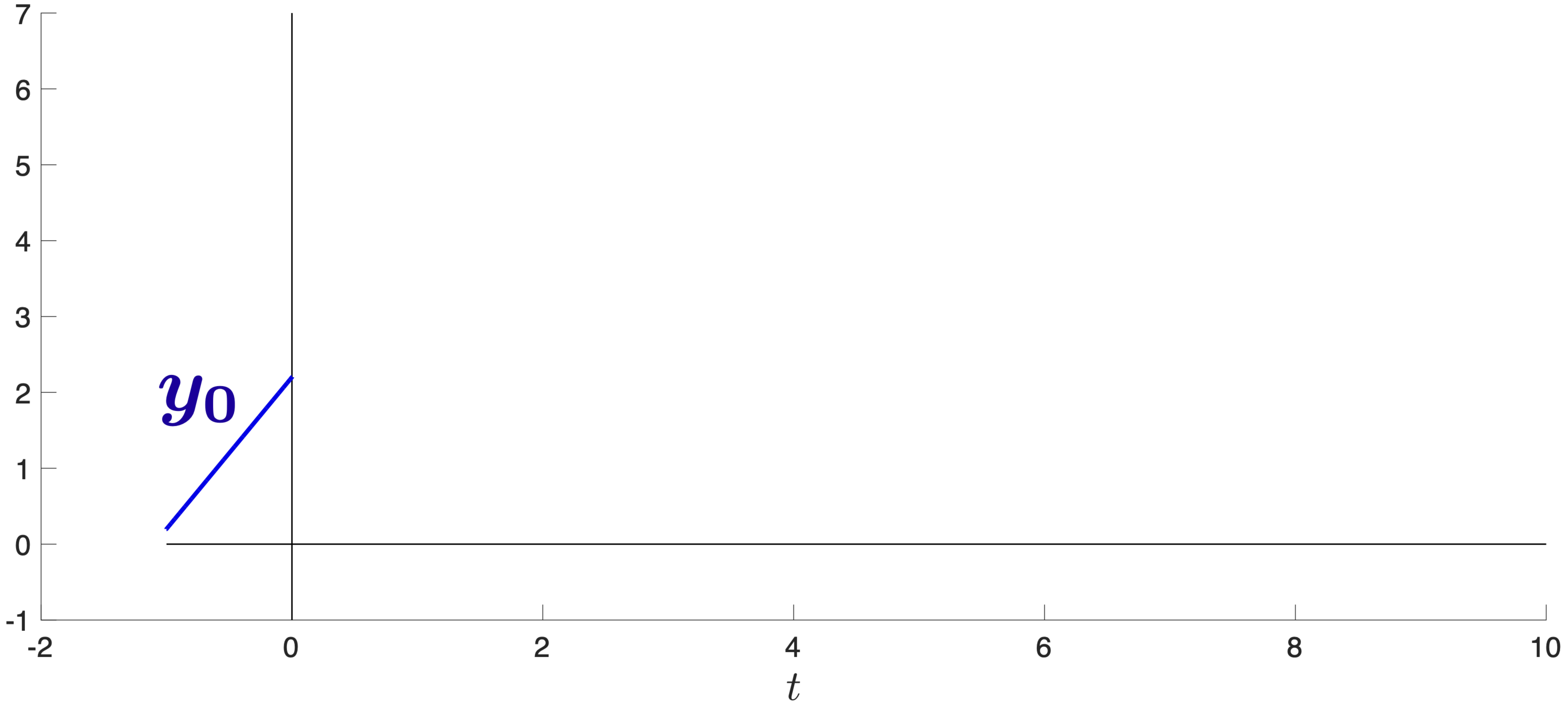
(the step map)

$$\boxed{F(y_0) \overset{\text{def}}{=} y_1}$$

Delay equations as infinite dimensional dynamical systems

$$\boxed{y'(t) = -3y(t-1)(1+y(t))}$$

(Wright's equation)



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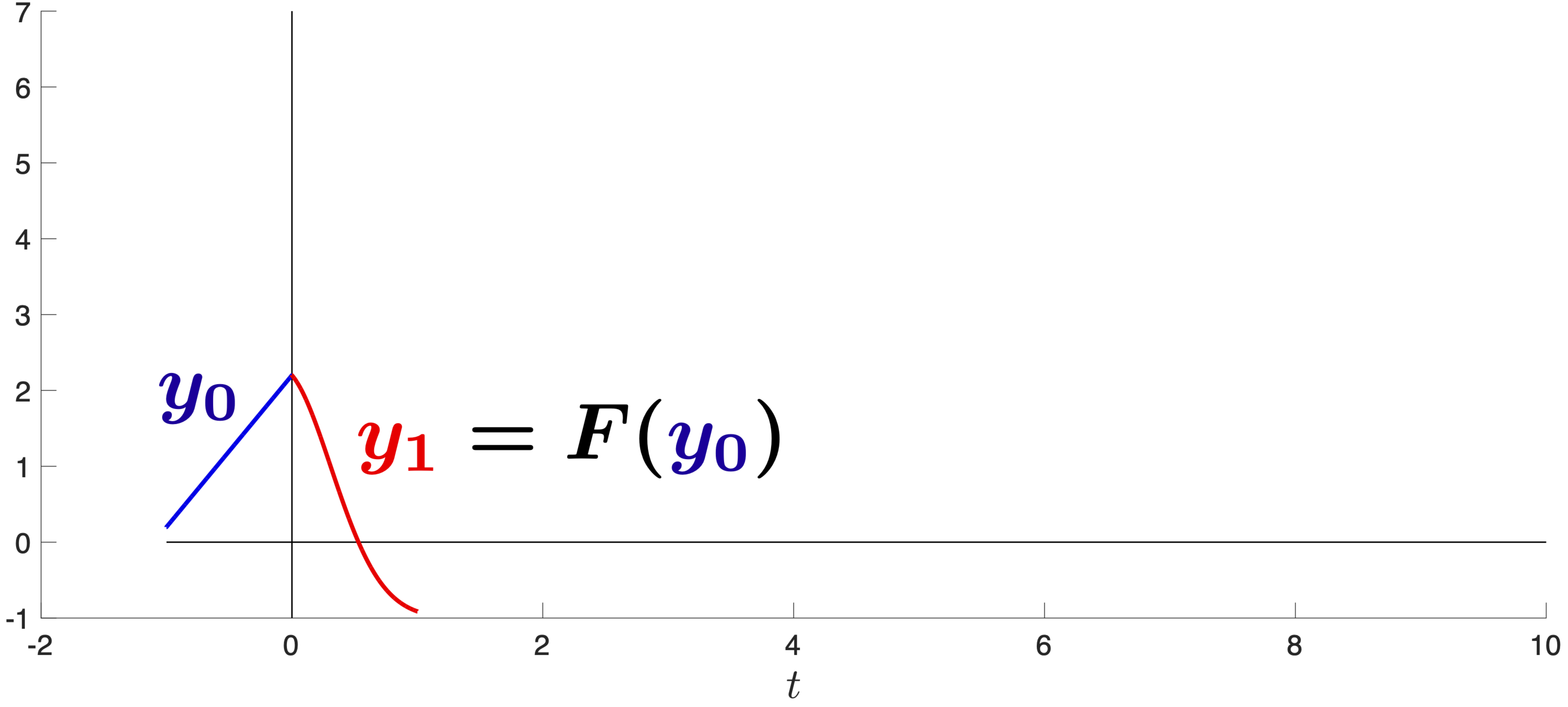

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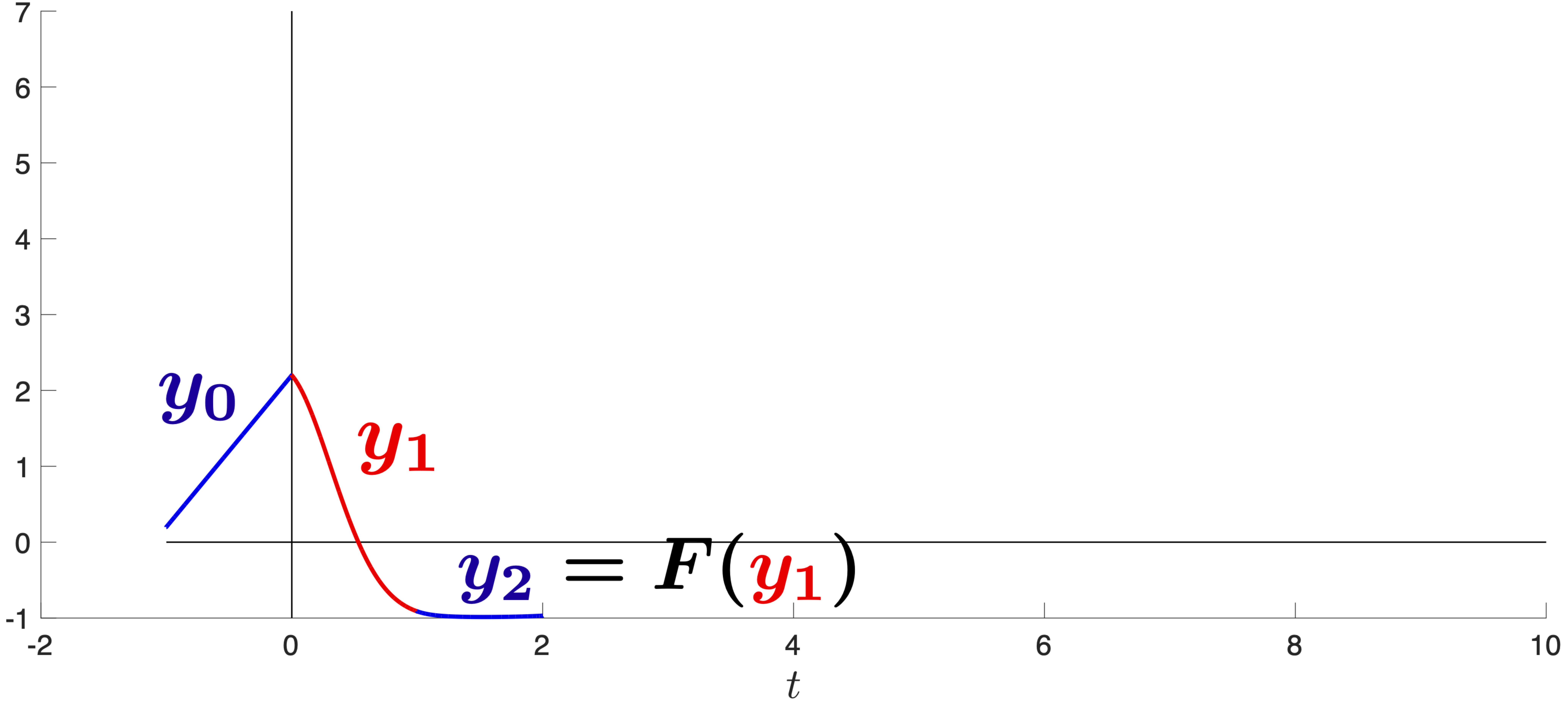

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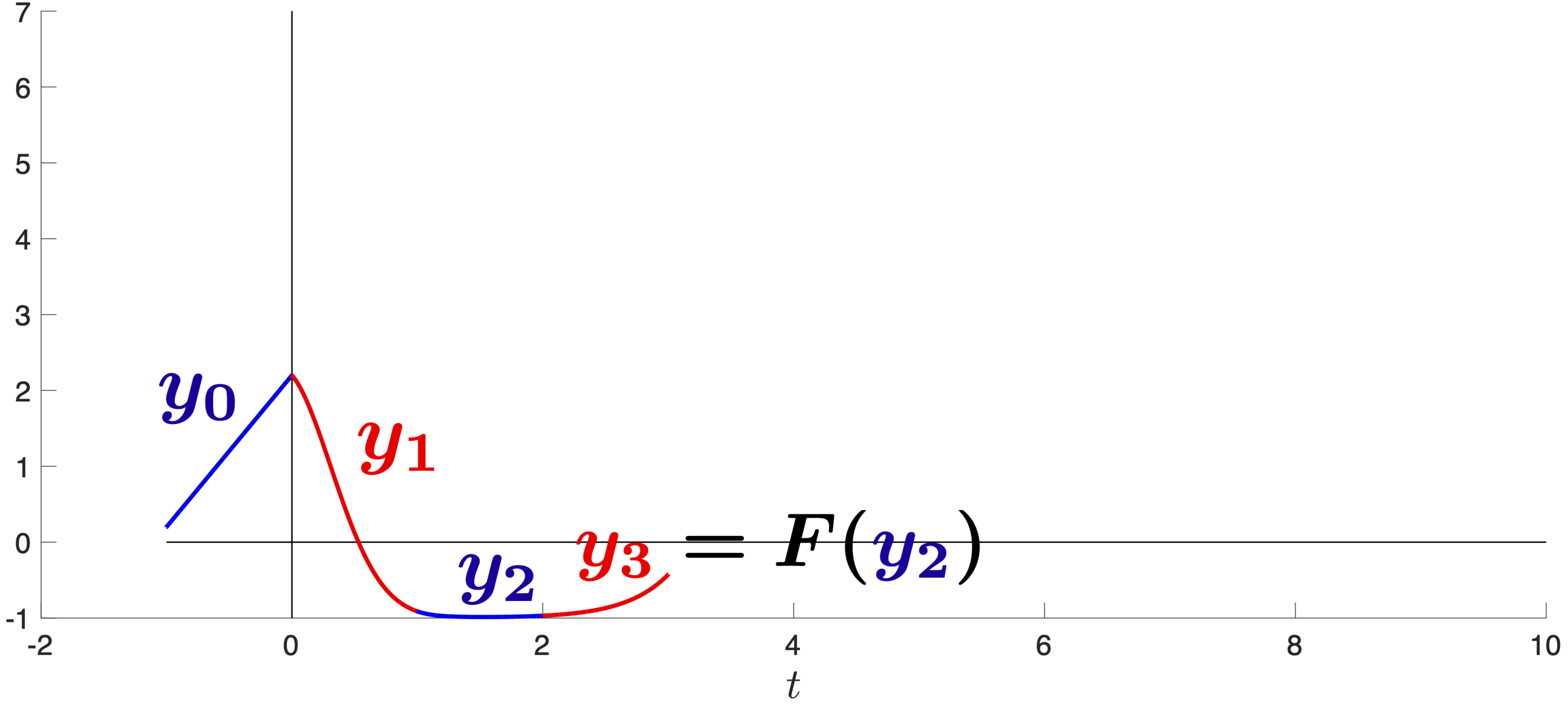

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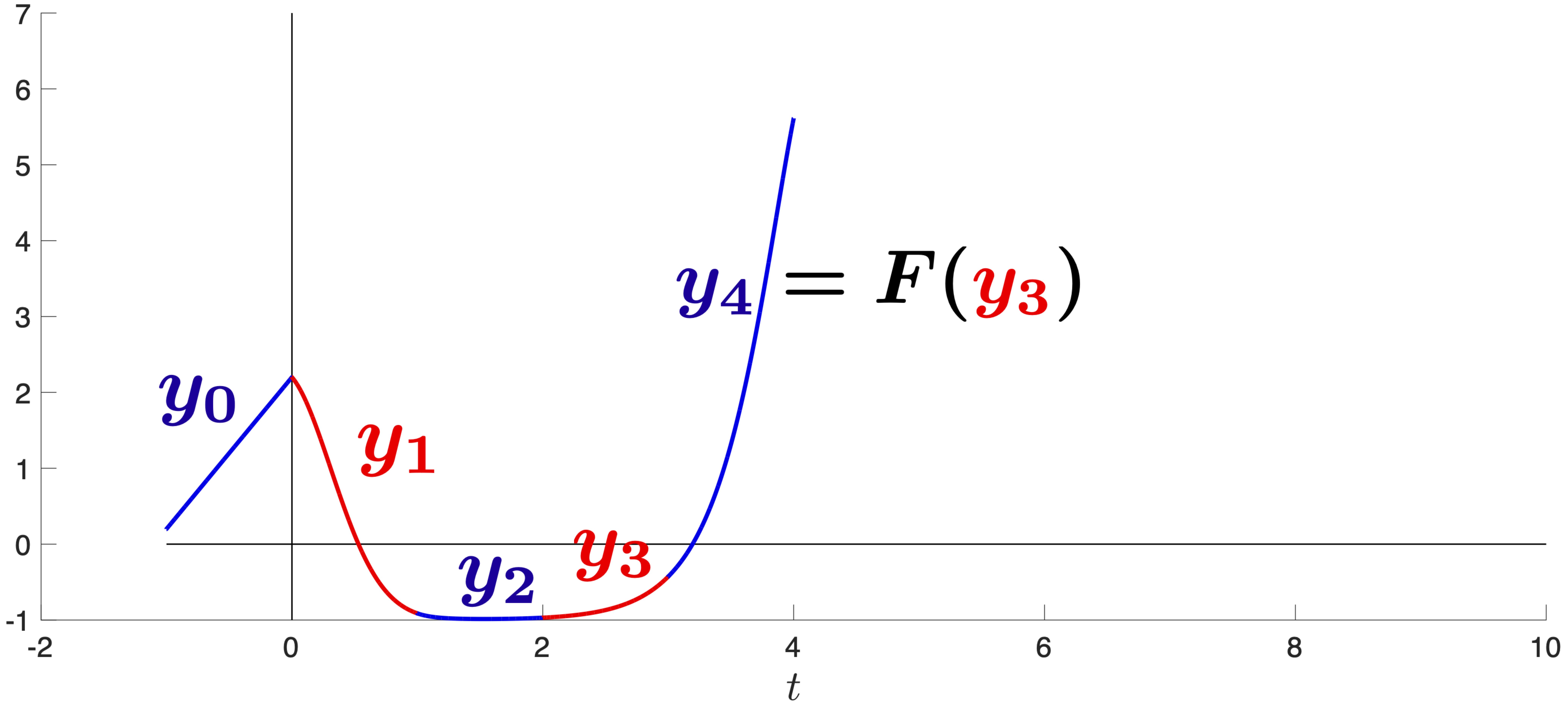

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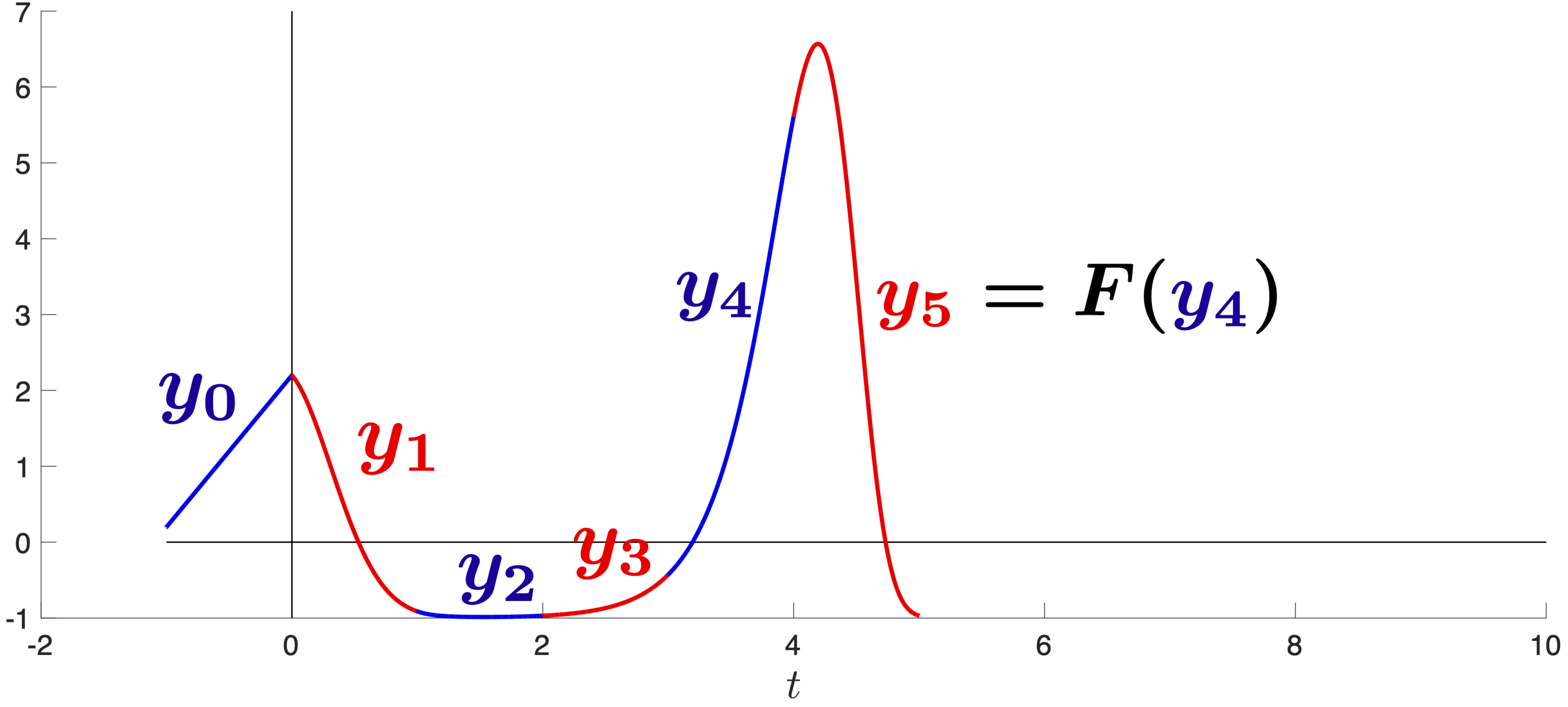

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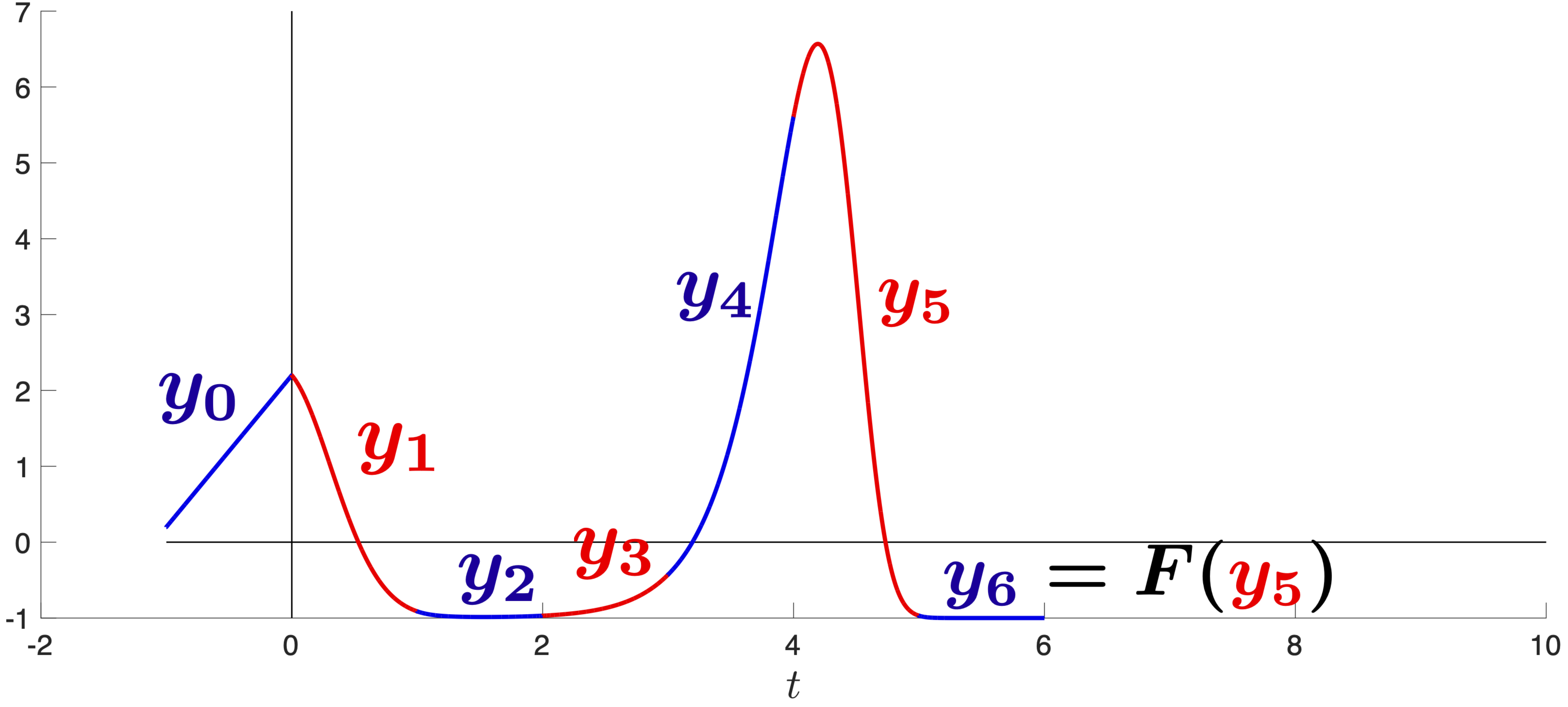

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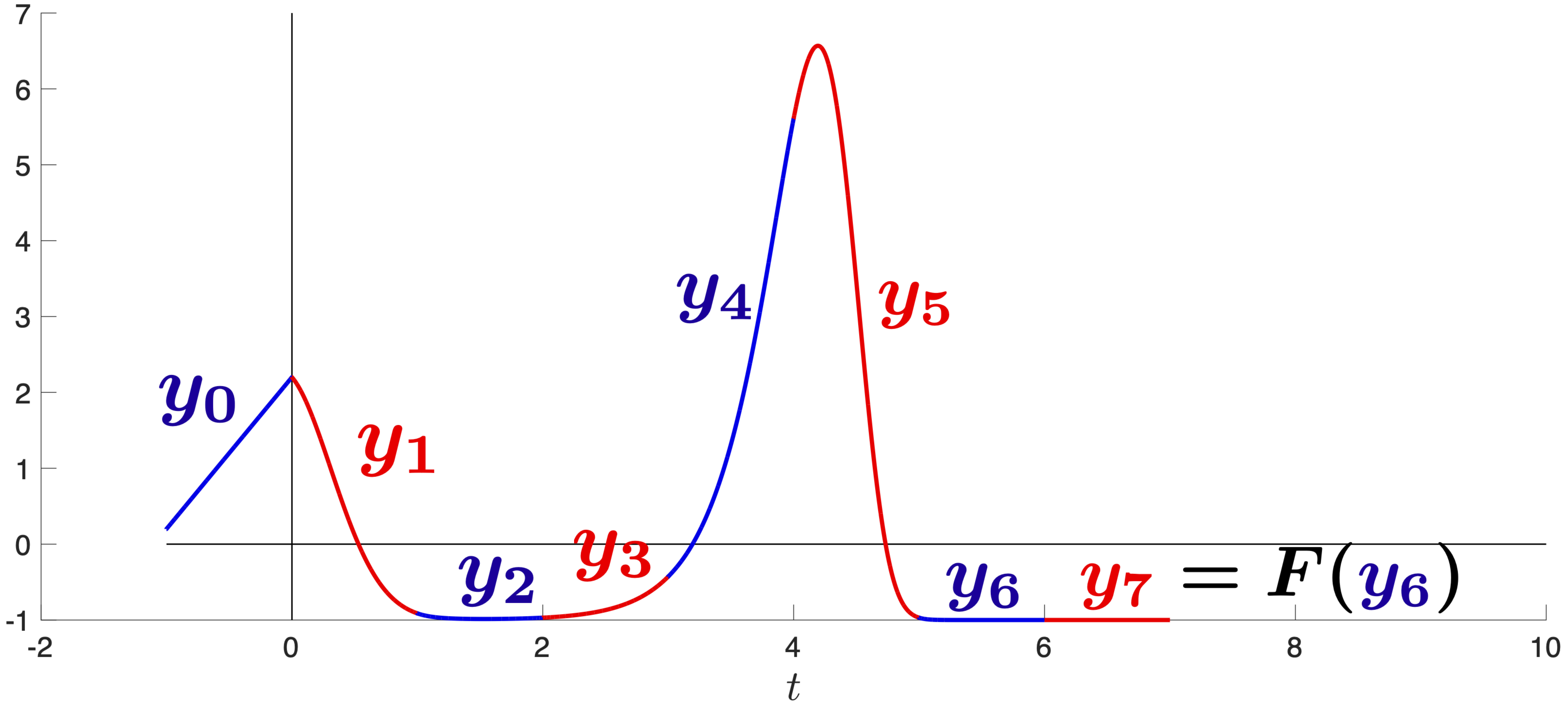

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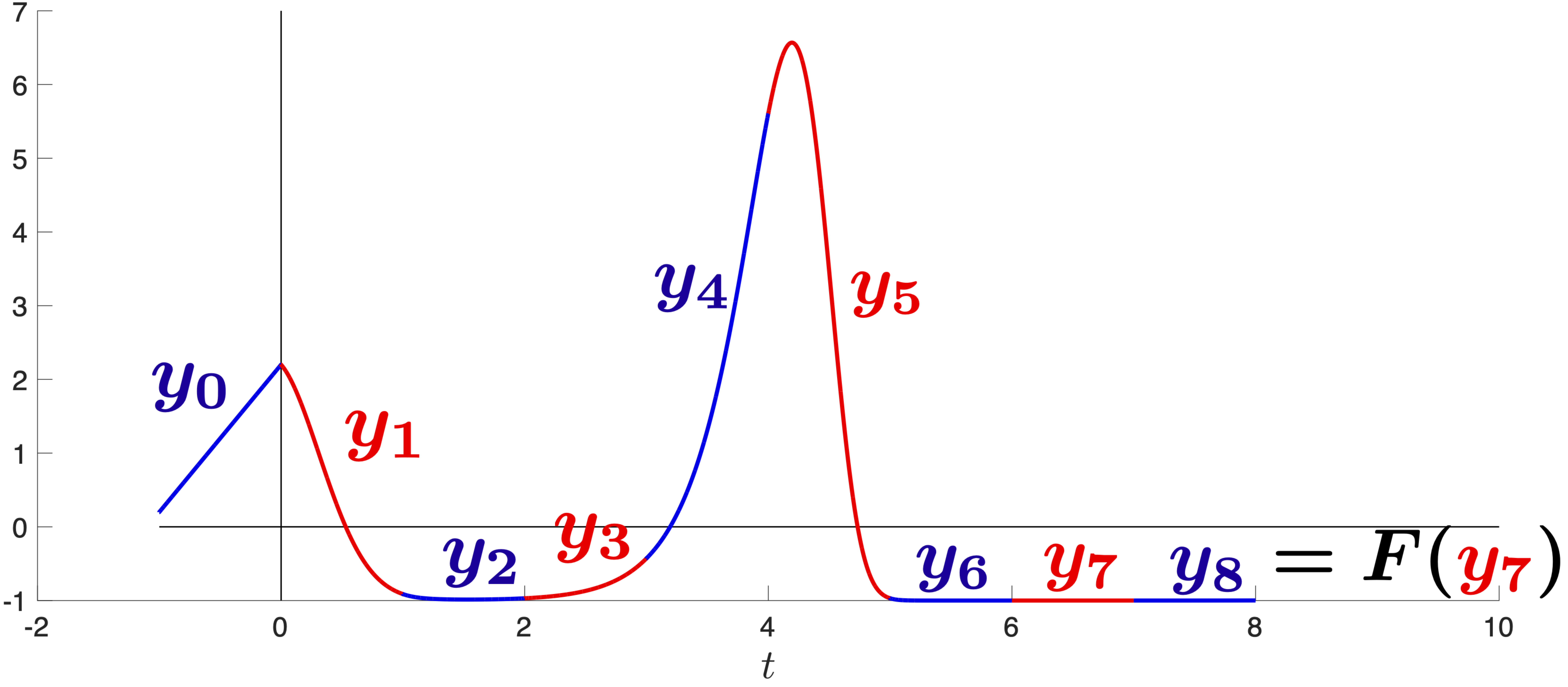

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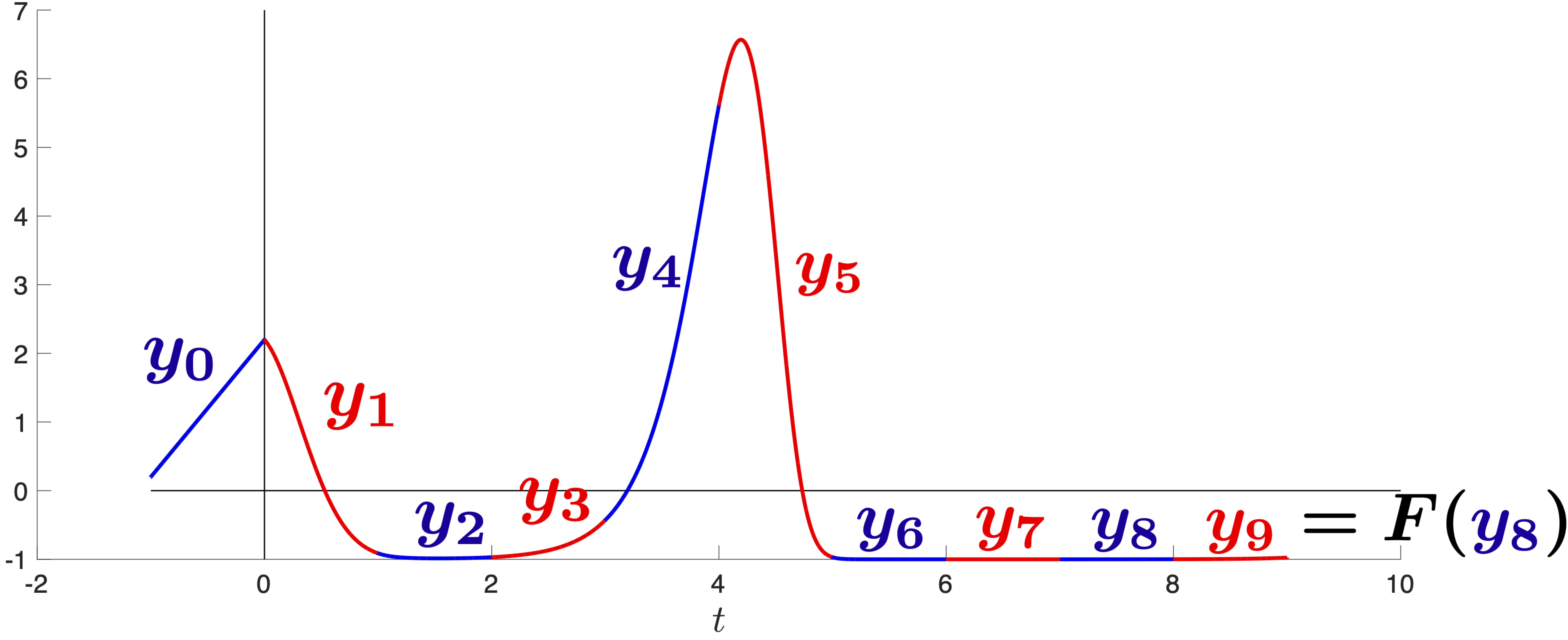

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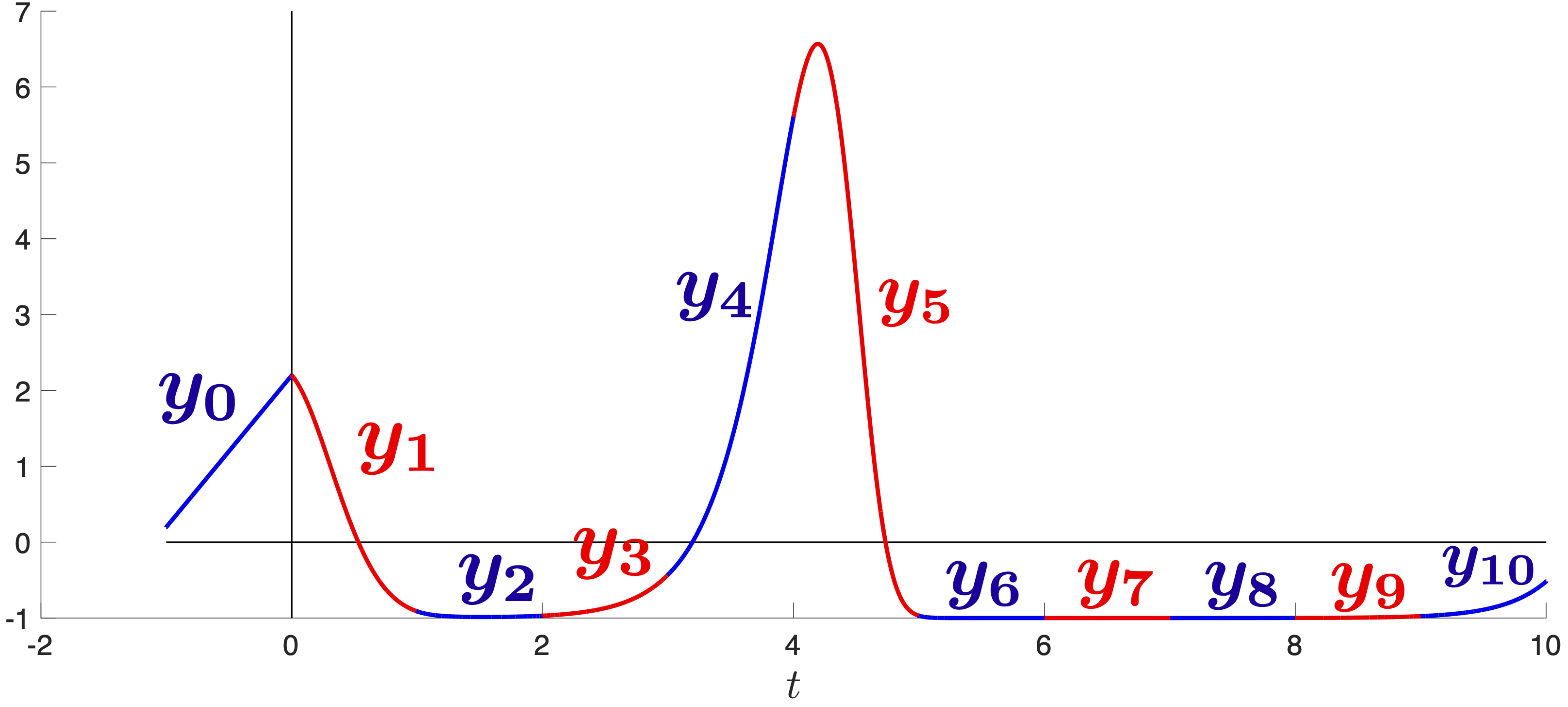

Computer-assisted proofs in delay equations with constant delays

- Periodic solutions (van den Berg, Church, Groothede, Kiss, L., Minamoto, Mireles-James, Nakao, Oishi, Szczelina, Zgliczyński)
- Wright's conjecture (van den Berg, Banhelyi, Csendes, Jaquette, Krisztin, Neumaier)
- Jones' conjecture (Jaquette, L., Mischaikow)
- Step maps via Chebyshev series (L. & Mireles-James)
- Stability, Hopf bifurcations (Church, L., Mireles-James)
- Unstable manifolds of equilibria & periodic orbits (Hénot, L. & Mireles-James)
- Chaos in constant delay perturbations of ODEs (Szczelina & Gierzkiewicz)
- Towards chaos in Mackey-Glass.....

State-dependent delay equations: motivation and challenges

An example of a scalar **state-dependent delay equation** (SDDE) is given by

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with $\mathcal{G}$ smooth and the delay $\tau(u(t))$ depending on the current state.

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A few places where SDDEs arise:

- **Electrodynamics**: charged particles interact via signals propagating at finite speed. In this case, the delay = distance / c.
- **Population dynamics**: maturation / reproduction times depend on density.
- **Physiology**: differentiation delays vary with hormone or blood-cell concentrations.
- **Engineering**: position-dependent tool speed in turning / milling.

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References: Walther (2003), Hartung, Krisztin, Walther & Wu (2006), Krisztin & Walther (2017)

Periodic solutions of SDDEs

Analytical features that fail to transfer from constant-delay DDEs

- Periodic solutions of analytic constant-delay DDEs are themselves analytic (Nussbaum, 1973)
- For SDDEs, analyticity vs. non-analyticity is much more subtle (Mallet-Paret & Nussbaum, 2014)

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Computational/numerical approaches for periodic solutions

- Hopf bifurcations for SDDEs (Calleja, Humphries & Krauskopf, 2017)
- Parameterization method for SDDE perturbations of ODEs (de la Llave, Gimeno & Yang, 2021)
- Projection of SDDEs via Lagrange-Chebyshev interpolation (Mireles-James, Motta & Naudot, '25)
- Picard operator & multiple-shooting method (Corbet & Naudot, 2025, 2026)
- Convergence of pseudo-spectral collocation (Ando & Sieber, 2025)

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Computer-assisted proofs (CAPs) in SDDEs

- Validated integration of SDDEs (Church, 2022)
- Periodic solutions of SDDE perturbations of ODEs (Gimeno, L. Mireles-James & Yang, '23)

Our main goal

We propose a constructive, computer-assisted method that

(A) produces a **rigorous proof of existence** of periodic solutions in *genuinely* state-dependent delay equations

(B) does **not rely** on perturbative reductions to ODEs or DDEs with constant delays

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Ingredients:

- Recast the problem as $\mathbb{F}(x) = 0$ in a low-regularity Banach space (C^1_{per})
- Fourier expansion of candidate periodic orbits
- Rigorous enclosure of Fourier coefficients of composition terms via **contour integrals**, **FFT** with interval arithmetic, and the **discrete Poisson formula**
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For SDDEs, the main difficulty: bounding terms of the form $u(t - \tau(u(t)))$ **rigorously**.

- Renormalization (Lanford, Breden, Gonzalez, Mireles-James, Burbanks, Pickett, Rahman)

A Newton–Kantorovich theorem

Theorem (Newton–Kantorovich)

Let $(X, \|\cdot\|_X)$ *be a Banach space,* $\mathbb{F} : X \to X$ *of class* C^1*,* $\bar{x} \in X$ *and* $\mathbb{A} \in B(X)$*. Fix* $r_* > 0$ *and assume that the non-negative constants* Y, Z_1, Z_2 *satisfy*

$$\begin{aligned} \|\mathbb{A}\mathbb{F}(\bar{x})\|_X &\leq Y \\ \|I - \mathbb{A}D\mathbb{F}(\bar{x})\|_{B(X)} &\leq Z_1 \\ \|\mathbb{A}\big[D\mathbb{F}(x) - D\mathbb{F}(\bar{x})\big]\|_{B(X)} &\leq Z_2\|x - \bar{x}\|_X, \qquad \|x - \bar{x}\|_X \leq r_*. \end{aligned}$$

If there exists $r_0 \in (0, r_*]$ *with*

$$Y + Z_1 r_0 + \tfrac{1}{2} Z_2 r_0^2 \leq r_0, \qquad Z_1 + Z_2 r_0 < 1,$$

then $\mathbb{F}$ *has a* **unique zero** $\widetilde{x} \in \overline{B_{r_0}(\bar{x})} \subset X$*.*

A zero-finding problem for periodic orbits

Consider the forced scalar toy SDDE model

$$\boxed{u'(t) = g(u)(t) \overset{\text{def}}{=} -u(t) - c_1 u\big(t - 1 - c_2 u(t)\big) + \cos(t) + \sin(t)}$$

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Define $\mathbb{F} : X \to X$ by

$$\mathbb{F}(\beta, u) \overset{\text{def}}{=} \begin{pmatrix} \mathbb{M}g(u) \\ u - \beta - \mathbb{L}_1 g(u) \end{pmatrix}.$$

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Theorem (Zeros of $\mathbb{F}$ are periodic solutions)

$\mathbb{F} : X \to X$ is well-defined and Fréchet differentiable. If $\mathbb{F}(\beta, u) = 0$, then u is a smooth 2π-periodic solution of the SDDE.

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Idea: $\mathbb{M}g(u) = 0$ forces $g(u)$ to have zero average, so $\mathbb{L}_1 g(u)$ is a genuine antiderivative; the second equation gives $u' = g(u)$.

Finite-dimensional projection, numerics, approximate inverse & bounds

Fix $N \in \mathbb{N}$ and let $u \in C^1_{\mathrm{per}}([-\pi,\pi])$, which admits a Fourier series $u(t) = \sum_{n\in\mathbb{Z}} U_n e^{int}$.

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Decompose $u = \pi^{\leq N} u + \pi^{>N} u$, where $\pi^{\leq N} u(t) \stackrel{\text{def}}{=} \sum_{|n|\leq N} U_n e^{int}$.

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Step 1 (numerical): apply Newton's method in finite precision to $\pi^{\le N}\mathbb{F}(\pi^{\le N} x) = 0$ to produce an approximate zero $\bar{x} = (\bar{\beta}, \bar{u}) \in X^{\le N} \stackrel{\text{def}}{=} \mathbb{R} \times \pi^{\le N} C^1_{\text{per}}([-\pi,\pi])$.

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Step 2 (approximate inverse): $D\mathbb{F}(\bar{x}) = I + K$ with K **compact**. Write

$$\mathbb{A} \; = \; \mathbb{A}^{\leq N} \; + \; \pi^{>N},$$

with $\mathbb{A}^{\leq N} : X^{\leq N} \to X^{\leq N}$ a numerical inverse of $\pi^{\leq N} D\mathbb{F}(\bar{x}) \pi^{\leq N}$.

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Step 3: bound Y, Z_1, Z_2 **rigorously in Fourier space**:

- Y **bound**: $\|\mathbb{A}\mathbb{F}(\bar{x})\|_X \leq Y$ using $\|u\|_\infty \leq \|U\|_{\ell^1}$ (needs the Fourier coefficients of the composition)
- Z_1 **bound**: uses a Lebesgue-constant estimate $\|\pi^{>N} h\|_\infty \leq \frac{1+\Lambda_N}{N} \|h'\|_\infty$
- Z_2 **bound**: Lipschitz bound on the composition nonlinearity

The composition challenge: Fourier coefficients of $\bar{u}(t-1-c_2\bar{u}(t))$

Given a trigonometric polynomial $\bar{u}(t) = \sum_{|n|\leq N} \bar{U}_n e^{int}$, enclose the Fourier coefficients V_n of

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Two-step enclosure:

- **(Tail)** for $|n| \ge N_{\texttt{tail}}$: prove $|V_n| \le C/\bar{\nu}^{|n|}$ via **contour integrals**
- **(Finite)** for $|n| \le N_{\texttt{tail}}$: compute V_n by **FFT in interval arithmetic** + rigorous aliasing control via the **discrete Poisson formula**

Tail bound via contour integrals

Shifting the contour of integration to $\mathbb{R} - i\bar{\rho}$, from analyticity on $\mathcal{S}_{\bar{\rho}}$ and using 2π-periodicity:

$$V_n = \frac{1}{2\pi}\int_0^{2\pi} v(t)e^{-int}\,dt = \frac{e^{-n\bar{\rho}}}{2\pi}\int_0^{2\pi} v(t - i\bar{\rho})e^{-int}\,dt, \qquad n \geq 0.$$

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Therefore

$$\boxed{|V_n| \;\leq\; \frac{C}{\bar{\nu}^{|n|}}, \qquad n \in \mathbb{Z}} \qquad C \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} |v(t-i\bar{\rho})|\,dt \quad \text{(sym. for } n<0\text{).}$$

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The constant C is computed rigorously by evaluating $v(t-i\bar{\rho}) = \bar{u}(t-i\bar{\rho}-1-c_2\bar{u}(t-i\bar{\rho}))$ on a grid using **FFT in interval arithmetic**: for $t \in [t_k, t_{k+1}]$ with $t_k = \frac{2k\pi}{N_{\mathrm{FFT}}}$, for $k = 0,\ldots,N_{\mathrm{FFT}}-1$,

$$\bar{u}(t-i\bar{\rho}) \;=\; \sum_{|n|\leq N} \left(\bar{U}_n e^{n(\bar{\rho}+i\delta)}\right)e^{int_k}, \qquad \boldsymbol{\delta} = [0, 2\pi/N_{\mathrm{FFT}}] \quad \text{(interval)}.$$

Everything computed in **interval arithmetic**.

Finite coefficients via FFT and discrete Poisson formula

Choose $N_{\texttt{FFT}} \gg 2N_{\texttt{tail}}$. The inverse FFT of $\{v(t_k)\}$ gives $\bar{V}_n$, $|n| \leq N_{\texttt{FFT}}/2$.

Discrete Poisson summation formula:

$$\bar{V}_n = V_n + \sum_{j\geq 1}\left(V_{n+jN_{\texttt{FFT}}} + V_{n-jN_{\texttt{FFT}}}\right).$$

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$$\boxed{|V_n - \bar{V}_n| \;\le\; 2C\left(\tfrac{1}{\bar{\nu}^n} + \bar{\nu}^n\right)\frac{1}{\bar{\nu}^{N_{\texttt{FFT}}} - 1}, \qquad |n| \le N_{\texttt{tail}}} \qquad \text{(}\textbf{aliasing error}\text{)}.$$

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Summary: given $\bar{u}$ and parameters $(N, N_{\texttt{tail}}, N_{\texttt{FFT}}, \bar{\nu})$:

contour integral $\Rightarrow$ tail bound | **FFT** $\Rightarrow$ $\bar{V}_n$ | **discrete Poisson** $\Rightarrow$ aliasing bound.

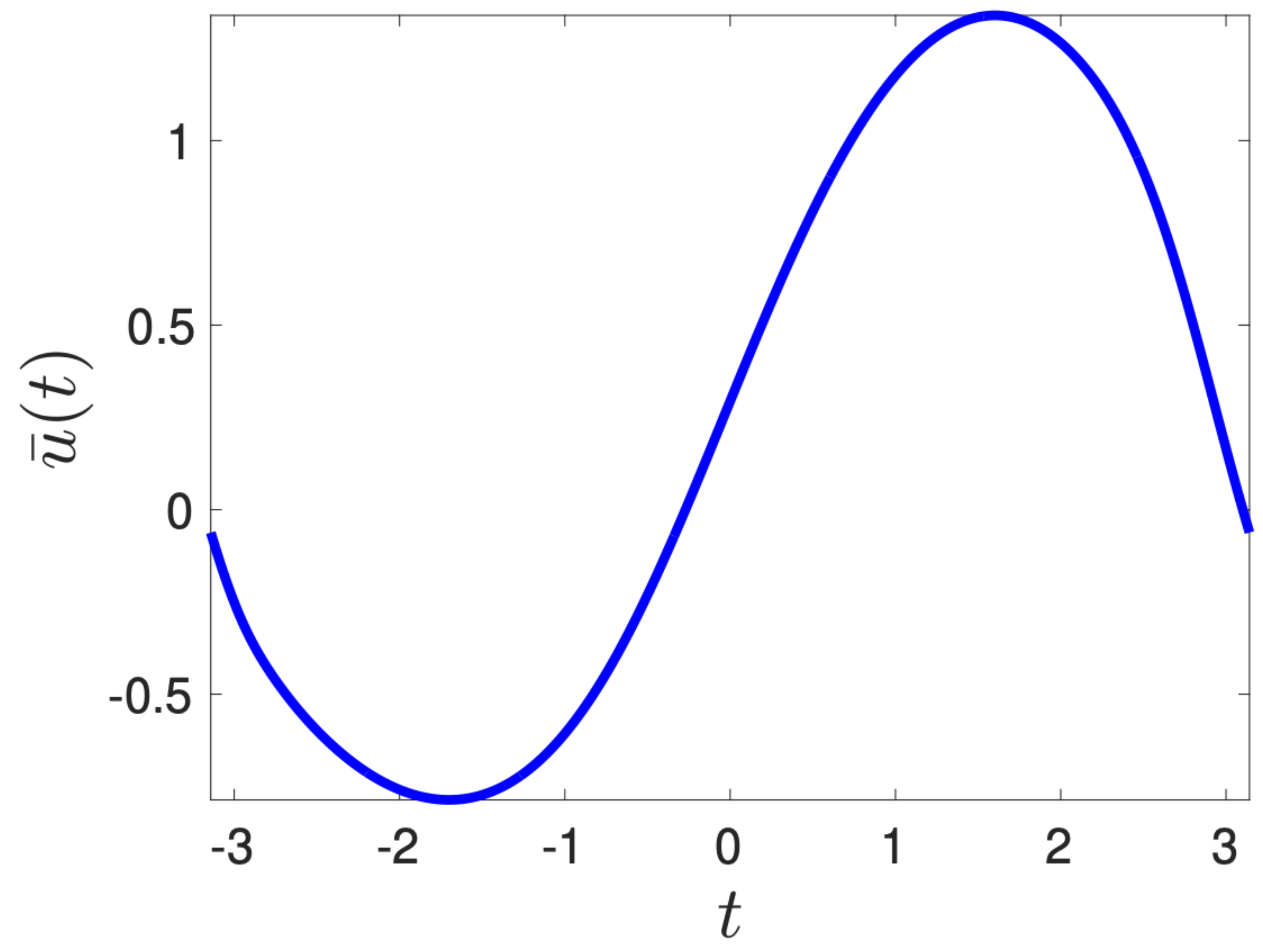
Example 1: Forced toy SDDE

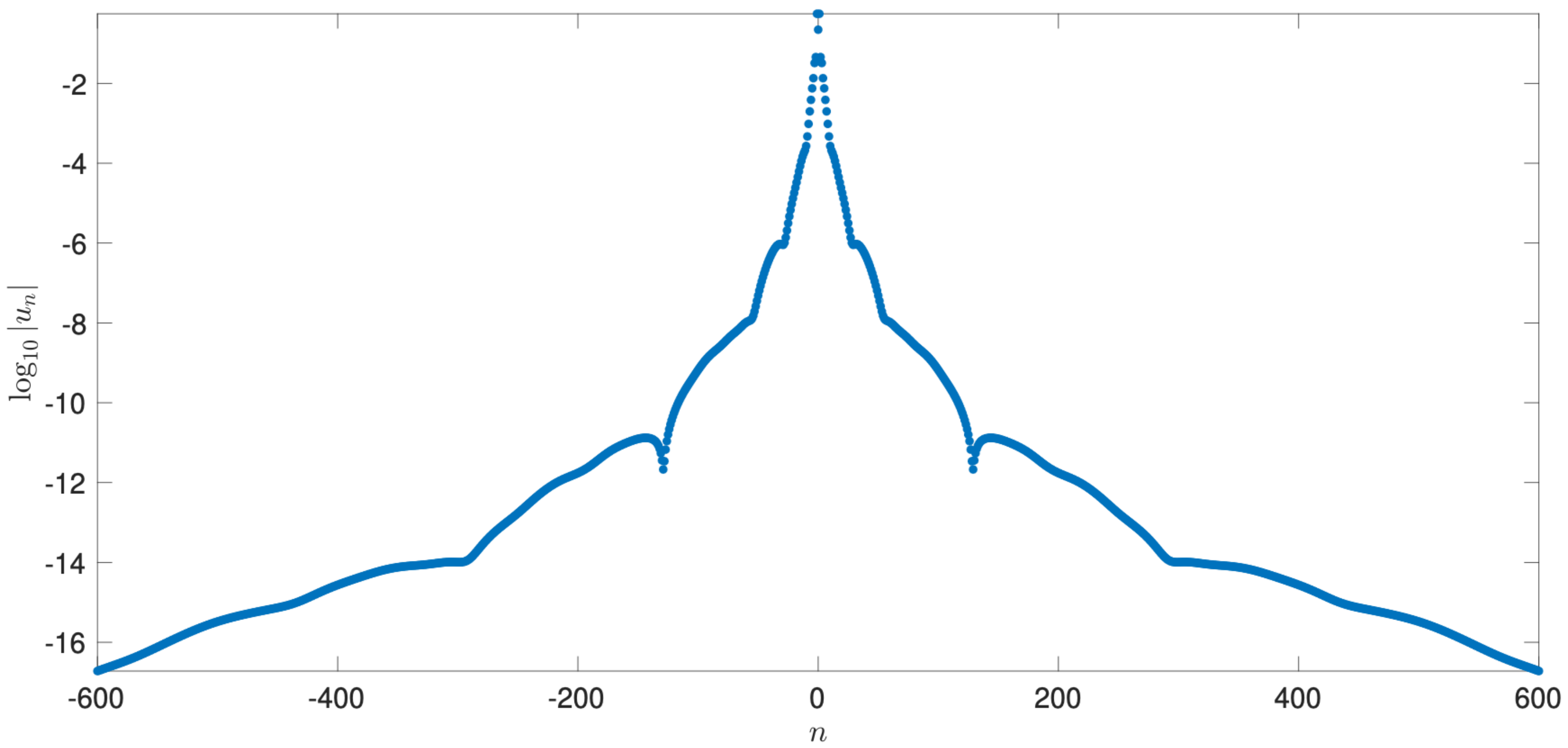
$$\boxed{u'(t) = -u(t) - 0.5\,u\big(t - 1 - 2u(t)\big) + \cos(t) + \sin(t)}$$

Fix $N = 140$, $\bar{\nu} = 1.1$, $N_{\texttt{tail}} = 669$, $N_{\texttt{FFT}} = 2^{12}$. The composition machinery yields $\boxed{C = 4.93 \times 10^{13}}$, and the Newton–Kantorovich bounds satisfy

$$Y = 1.59 \times 10^{-7}, \quad Z_1 = 0.667, \quad Z_2 = 298.1.$$

Proof successful with radius $r_0 = 4.75 \times 10^{-7}$.





This model serves as the prototype. The two next examples introduce the additional challenges of *unknown period* and *threshold delays*.

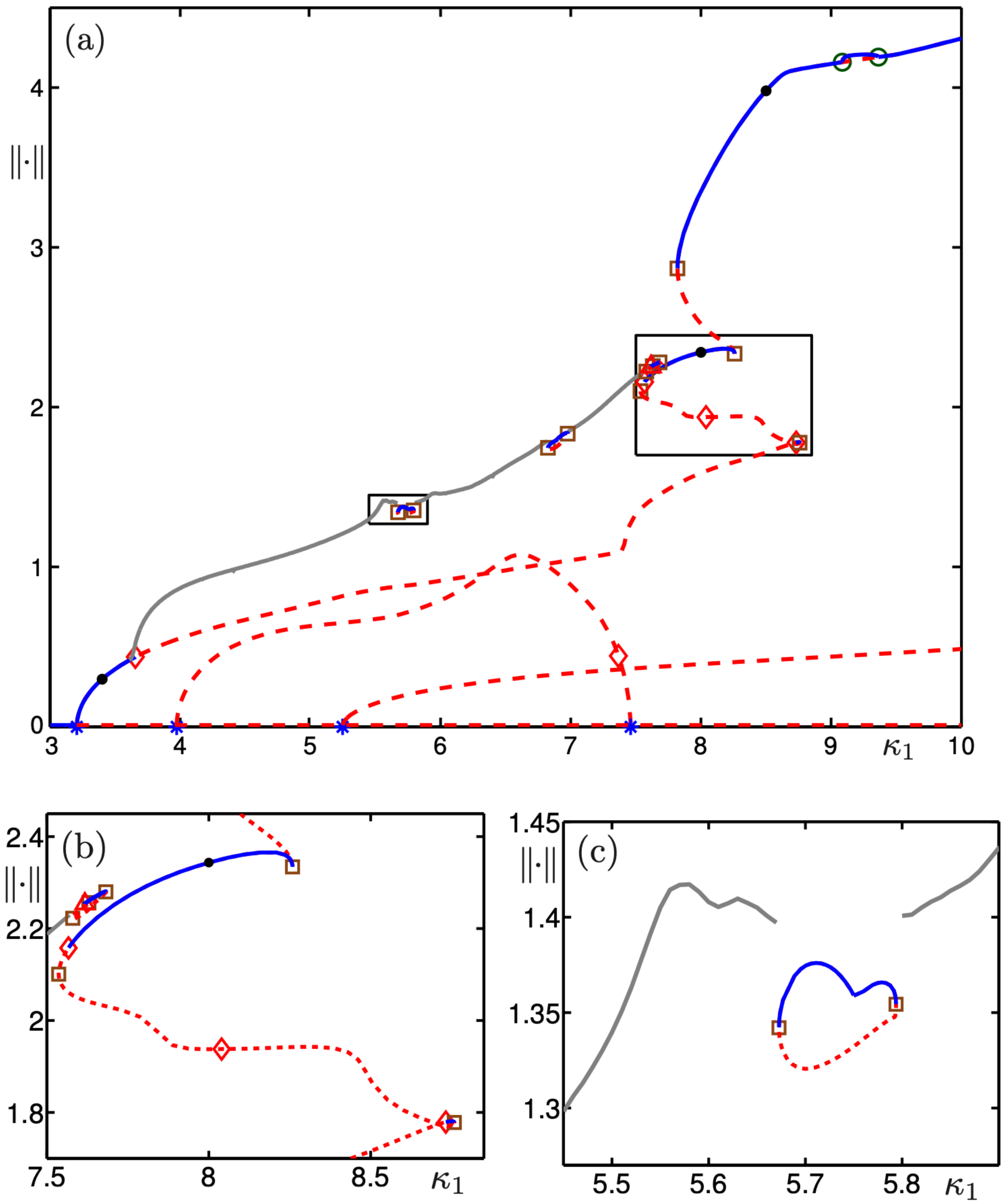
Example 2: Humphries, Calleja & Krauskopf (SIADS, 2017)

Scalar autonomous SDDE with **two state-dependent delays**:

$$u'(t) = -\gamma\, u(t) - \kappa_1\, u(t - a_1 - c_1 u(t)) - \kappa_2\, u(t - a_2 - c_2 u(t))$$

Parameters:

$$\gamma = 4.75,\ a_1 = 1.3,\ a_2 = 6,\ c_1 = c_2 = 1,\ \kappa_2 = 3.$$

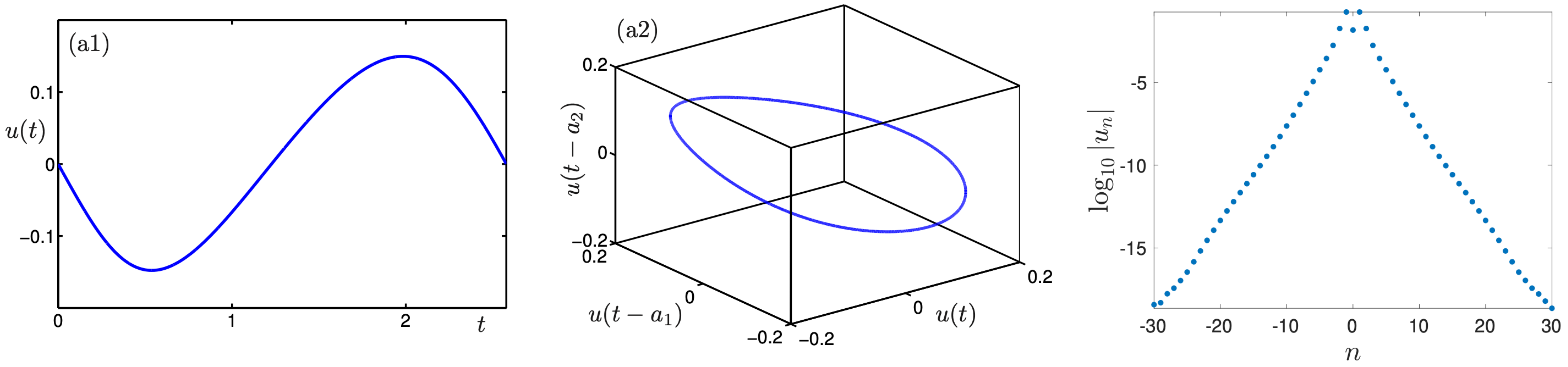


Calleja, Humphries & Krauskopf computed numerically a rich bifurcation diagram in $\kappa_1 \in (0, 4.75)$, revealing resonance phenomena between Hopf branches.

Our goal: prove rigorously the existence of periodic orbits on these branches.

Example 2: Unknown period $\Longrightarrow \omega$ becomes a variable

For $\kappa_1 = 3.4$, the numerical periodic orbit:



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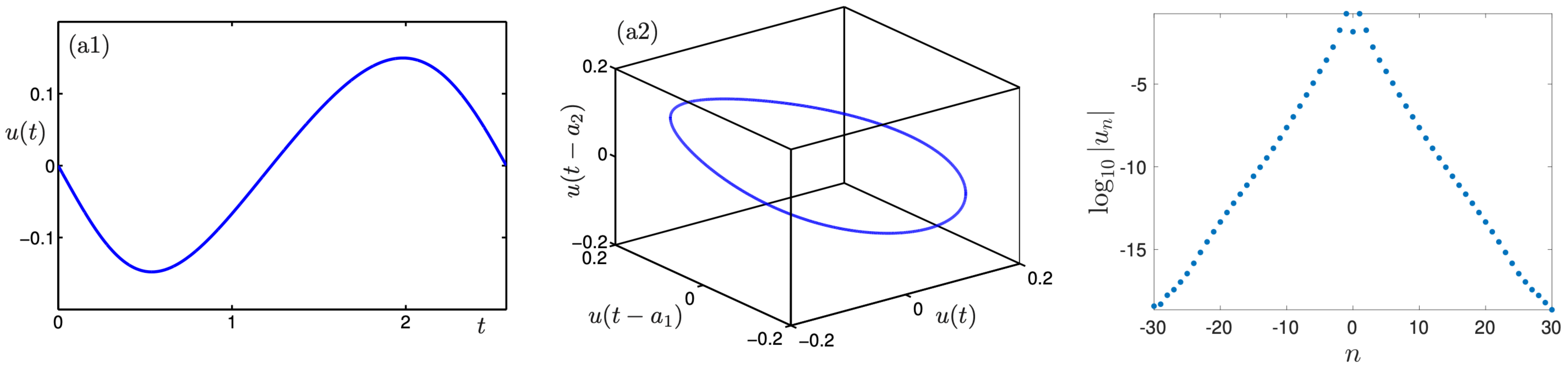

New feature: the period $2\pi/\omega$ is *unknown* $\Rightarrow$ ω becomes a variable.

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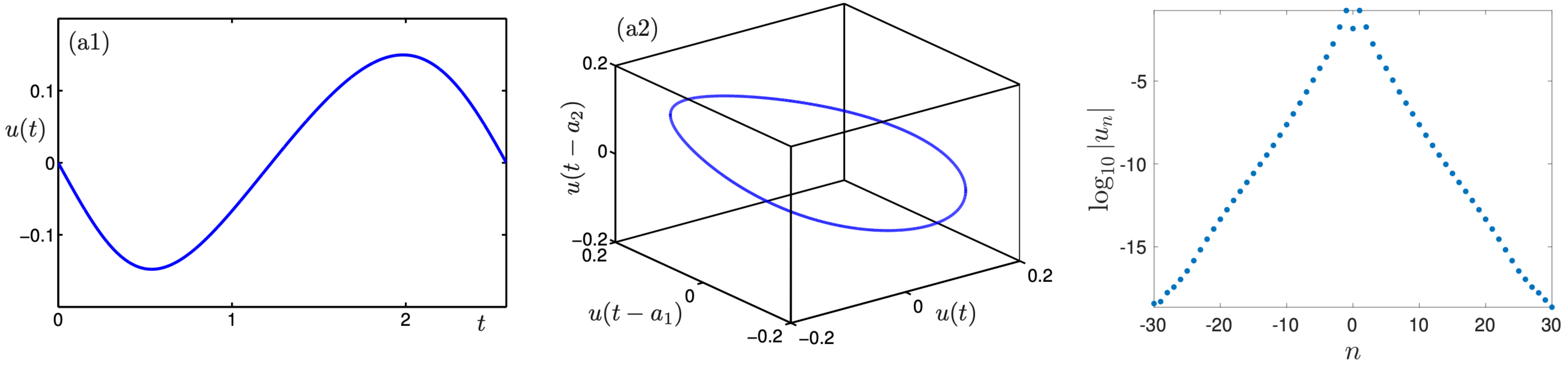

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Extended Banach space: $x = (\beta, u) \in \mathbb{R}^2 \times C^1_{\text{per}}([-\pi, \pi])$, with $\beta = (\beta_1, \beta_2) = (\omega,\, u(-\pi))$.

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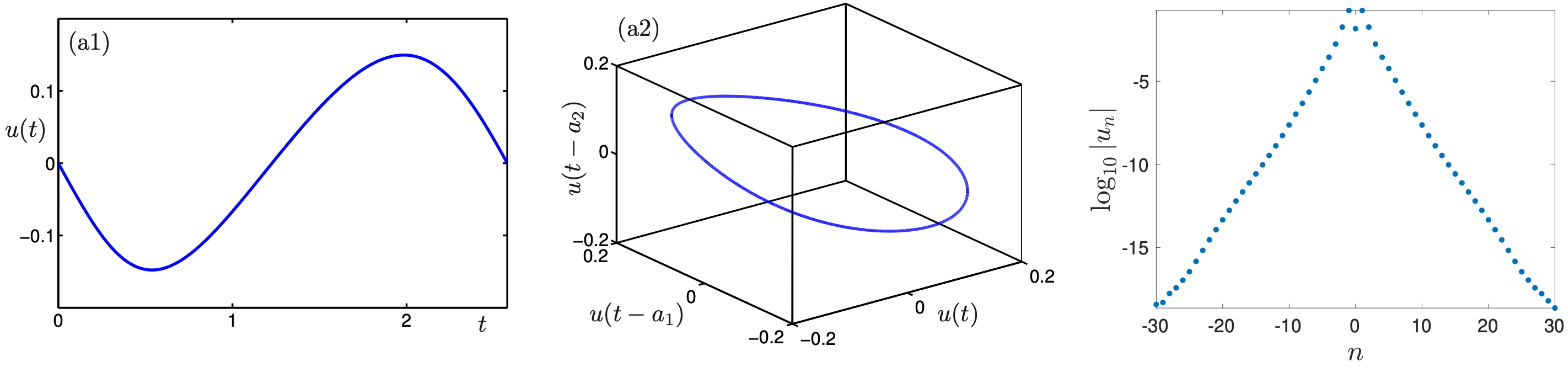

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Add a **phase condition** to break the translation invariance:

$$\xi_1(x) = (\pi^{\leq N} u)(0) - u_0, \qquad \xi_2(x) = \mathbb{M}g(\omega, u).$$

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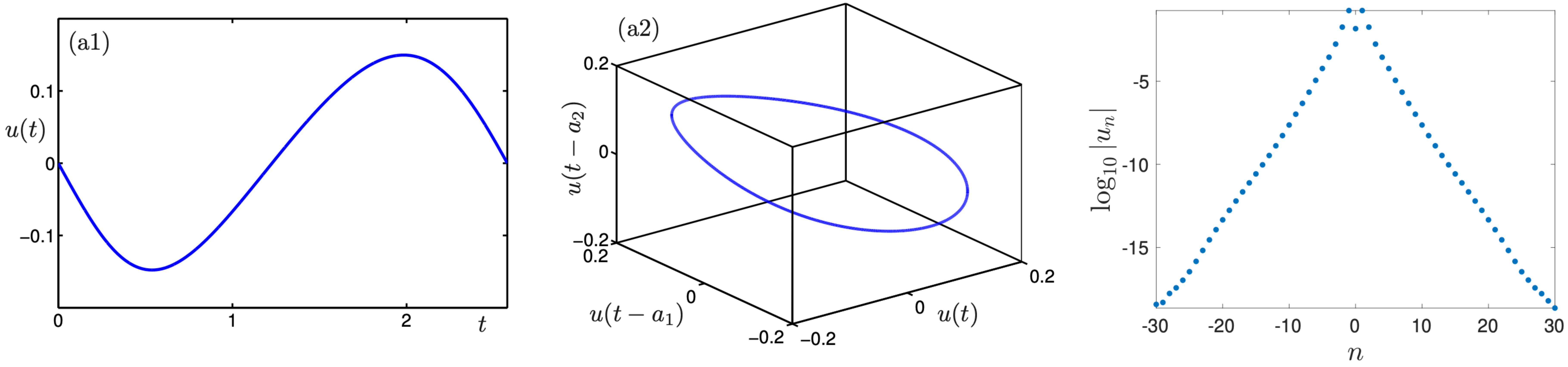

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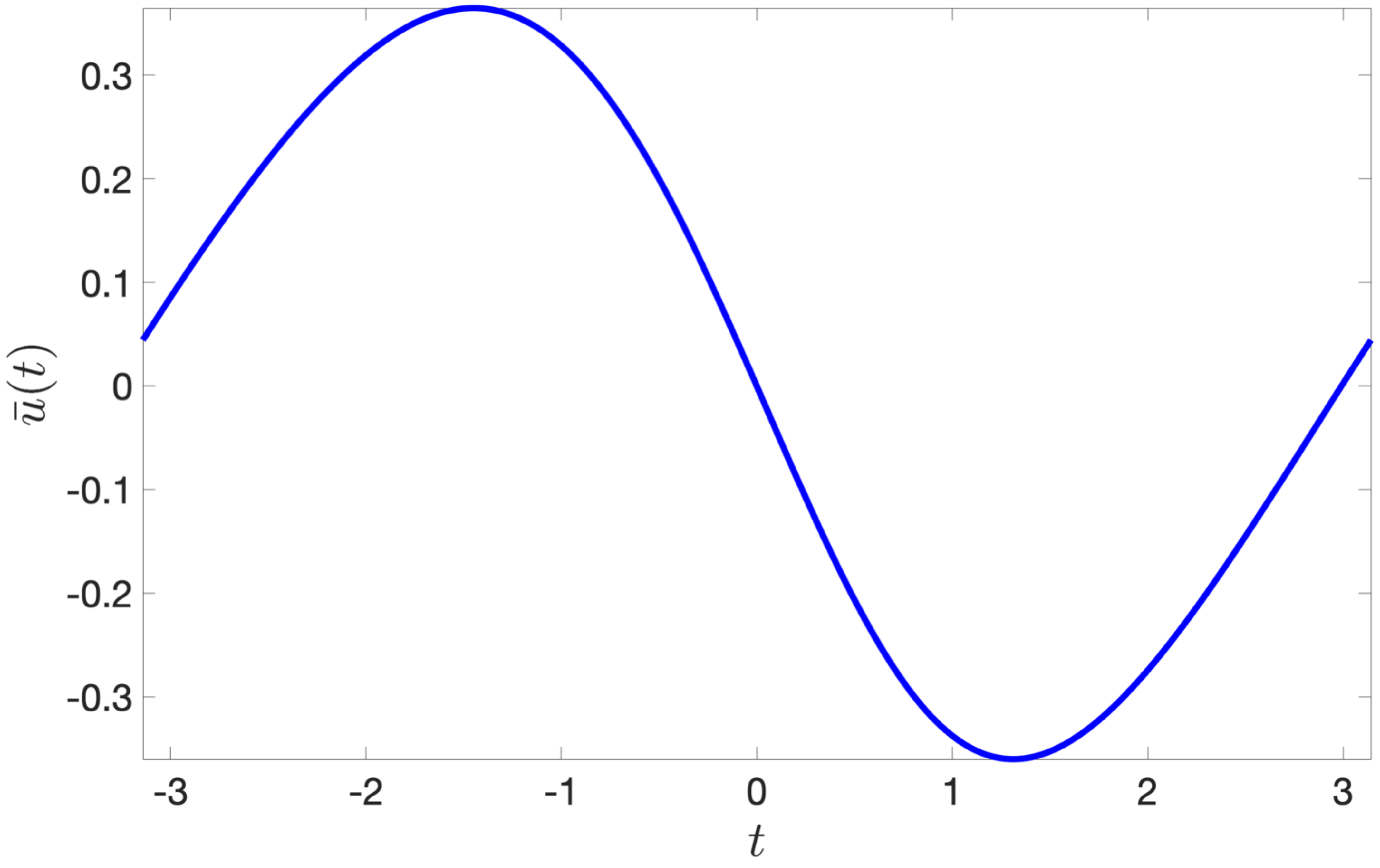
The map $\mathbb{F}$ (incorporating ξ) acts on $X = \mathbb{R}^2 \times C^1_{\text{per}}([-\pi,\pi])$, and the Newton–Kantorovich analysis proceeds as before.

Existence proof of a periodic solution

- $N = 800$
- $\|\mathbb{A}\mathbb{F}(\bar{x})\|_X \leq Y = 1.1 \times 10^{-9}$
- $\|I - \mathbb{A}D\mathbb{F}(\bar{x})\|_{B(X)} \leq Z_1 = 0.78$
- There exists a true solution $\widetilde{x} = (\widetilde{w}, \widetilde{\beta}, \widetilde{u}) \in \mathbb{R} \times \mathbb{R} \times C^1_{per}$ such that $\|\widetilde{u} - \bar{u}\|_{C^1} \leq 5.9 \times 10^{-9}$ with frequency $\widetilde{\omega} \approx \frac{2.6}{2\pi}$.



Example 3: Gedeon, Humphries, Mackey, Walther & Wang (2025)

Gene-regulation model for an autoregulated bacterial operon:

$$w'(t) = b\, e^{-\mu\tau(t)} \frac{v(w(t))}{v(w(t-\tau(t)))}\, q(w(t-\tau(t))) - \gamma\, w(t),$$
$$a = \int_{-\tau(t)}^{0} v(w(s+t))\, ds,$$

with Hill functions $v(x) = \dfrac{v^- \theta_v^m + v^+ x^m}{\theta_v^m + x^m}$ and $q(x) = \dfrac{q^- \theta_q^n + q^+ x^n}{\theta_q^n + x^n}$.

$w(t)$: protein concentration. γ: degradation rate. μ: dilution rate.

The delay $\tau(t)$ is **not clock-time** but the time for ribosomes/polymerases, moving at state-dependent velocity $v(w)$, to traverse the gene – a **threshold-type** state-dependent delay defined implicitly by the integral constraint.

Example 3: Numerical periodic orbits (Gedeon et al., 2025)

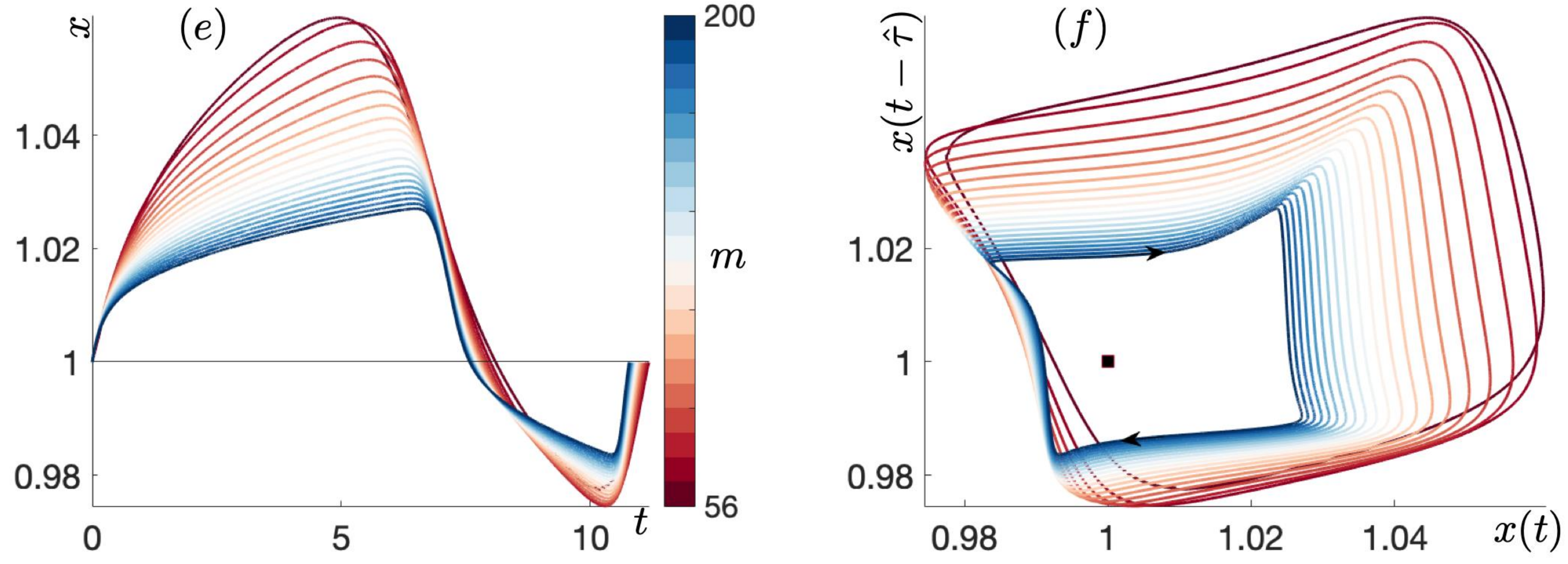


Figure 5.7. *Bifurcation diagram of (1.1)-(1.2) with $(g \leftrightarrow, v \downarrow)$ and parameters $\beta = 3, \mu = 0.3, g^- = g^+ = 1, \gamma = 1, \theta_v = 1, a = 3, v^- = 0.5$ and $v^+ = 0.2$. (a) The limiting case with v defined by (1.7). The stable steady state is shown as a green solid line, and the singular steady state as a black dashed line. (b) With a smooth velocity nonlinearity v defined by (1.3) with $m = 100$. Solid lines represent stable objects including the stable steady state (in green), and envelope of the periodic orbit (in red and blue). Dashed lines represent unstable steady states which have two eigenvalues with positive real part (in black). (c) As in (b) but with $m = 32$. (d) Two-parameter continuations in m and γ of the Hopf bifurcations (shown as solid curves) with the other parameters as above. The outermost curve of Hopf bifurcations is associated with the stability change seen in (b). The black dash-dotted line denotes $\gamma = \gamma_3 = 0.0333$ and the red dash-dotted line denotes $\gamma = \mu = 0.3$, the location of the Hopf bifurcations in the limiting case as $m \to \infty$. Note that $\gamma(\theta_v) = 0.2293 < \mu < \gamma_4 = 0.4959$. (e) Profiles of the stable periodic orbits from the Hopf bifurcations in (b) at $\gamma = 0.13$ and $m \in [56, 200]$ as indicated by the color map. (f) Projection of the phase space dynamics into the $(x(t), x(t-\hat{\tau}))$ plane at $\gamma = 0.13$ where $\hat{\tau} = \tau(\theta_v) = 60/7$. The arrow indicates the direction of the flow. The square marks the unstable steady state in the limiting case at the threshold.*

Numerically computed periodic orbits from (Gedeon, Humphries, Mackey, Walther & Wang, 2025)

Example 3: Reformulation as a system of SDDEs

Differentiating the threshold condition (Leibniz rule):

$$\tau'(t) = 1 - \frac{v(w(t))}{v(w(t-\tau(t)))}, \qquad a = \int_{-\tau(0)}^{0} v(w(s))\,ds \quad \text{(auxiliary condition).}$$

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$$\tau'(t) = 1 - \frac{v(w(t))}{v(w(t-\tau(t)))} + \eta.$$

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Theorem

If $(w, \tau, \eta) \in C^1_{\text{per}} \times C^1_{\text{per}} \times \mathbb{R}$ *solves the augmented system, then* $\eta = 0$.

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Proof: Multiplying by $v(w(t-\tau(t)))$ and rearranging,

$$\frac{d}{dt}\left(\int_{t-\tau(t)}^{t} v(w(s))\, ds\right) = \eta\, v(w(t-\tau(t))).$$

The LHS is periodic, so its integral over one period vanishes; v bounded away from 0 forces $\eta = 0$. $\square$

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The LHS is periodic, so its integral over one period vanishes; v bounded away from 0 forces $\eta = 0$. $\square$
Consequence: periodic orbits of the threshold SDDE $\leftrightarrow$ zeros of the augmented map at $\eta = 0$.

Example 3: Newton–Kantorovich setup for the threshold SDDE

The augmented phase space is

$$X = \mathbb{R}^4 \times C^1_{\mathrm{per}}([-\pi, \pi])^2, \quad \beta = (\eta, \omega, w(-\pi), \tau(-\pi)).$$

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The zero-finding map $\mathbb{F} : X \to X$ is augmented by

$$\xi(x) = \left(\omega \int_{-\tau(0)}^{0} v(w(s))\, ds - a,\ \ \pi^{\leq N} w(0) - 1,\ \ \mathbb{M}\, g(\omega, u) + \beta_1 e_2\right).$$

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Three additional layers of difficulty vs. Example 2:

- Genuinely vector-valued (w and τ simultaneously)
- Extra scalar constraint $\Rightarrow$ unfolding parameter η
- Nonlinear Hill functions composed with state-dependent arguments

(ongoing)

Conclusion and ongoing work

What we have done:

- A constructive, computer-assisted method for proving existence of periodic solutions in *genuinely* state-dependent delay equations.
- A Newton–Kantorovich framework on $X = \mathbb{R}^m \times (C^1_{\text{per}})^n$ with Fourier-coefficient bounds.
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Future directions:

- Adapt the strategy for the rigorous enclosure of the Fourier coefficients to algebraic decay
- Stability: rigorous Floquet multipliers, Hopf / period-doubling bifurcations
- Unstable manifolds via the parameterization method
- Applications: oscillatory gene regulation, N-body problem with finite-speed gravity, electrodynamics
- Non-recurrent dynamics & connecting orbits in the SDDE setting

References / Thanks!

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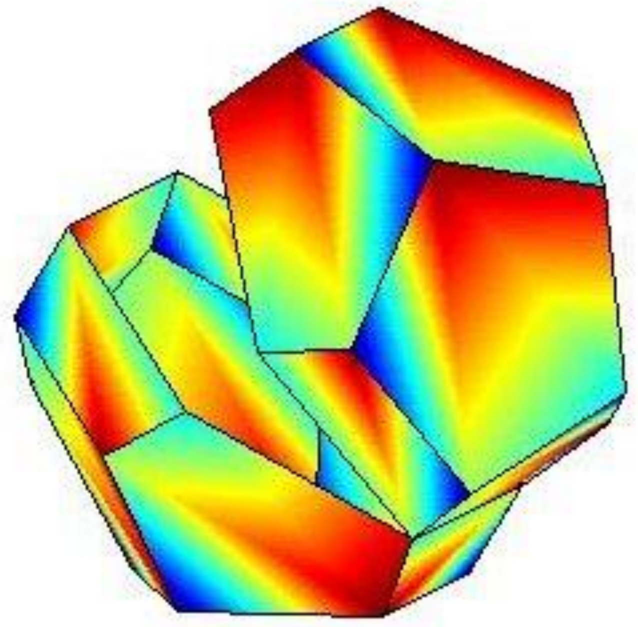

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Thank you for your attention!