Rigorous integration of differential inclusions in the CAPD library

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General problem

Definition

$$x'(t) = f(x(t), y(t)), \quad x(0) = x_0 \tag{1}$$

where

- $x \in \mathbb{R}^n$,
- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is C^1
- $y : \mathbb{R} \supset D \to \mathbb{R}^m$.

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$y(t)$ can be seen as:

- perturbation of an ODE,
- control,
- influence of higher order terms ...

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Assume that we have some knowledge about $y(t)$, for example

$$|y(t)| < \epsilon \text{ for } 0 \leq t \leq T.$$

We would like to find an rigorous enclosure for $x(t)$.

Motivation from differential inclusions

Differential Inclusion

$$x'(t) \in f(x(t)) = h(x(t)) + \epsilon(t), \tag{2}$$

where

- $h : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-vector field
- $\epsilon(t) \subset \mathbb{R}^n$.

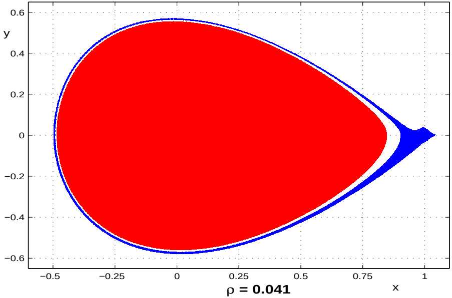
- How accurate are our simulations if we consider only simplified system?
- What is the worst case?

Motivations from Control Systems

$$x'(t) = f(x(t), u(t)) \qquad \text{where } u(t) \in U$$

U is a set of admissible control values.

- Ships capsizing.



Can ship be capsized by choosing appropriate sequence of parameters/controls values (such as wind, wave strength, rudder position)?

Motivations from Control Systems

$$x'(t) = f(x(t), u(t)) \qquad \text{where } u(t) \in U$$

U is a set of admissible control values.

- Ships capsizing.
- Space ships design.
  Can we be sure to reach given destination?
  What destinations cannot be reached?

Motivations from Control Systems

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- Ships capsizing.
- Space ships design.
  Can we be sure to reach given destination?
  What destinations cannot be reached?
- Correctness of numerical simulations
  Are our numerical simulations of controlled trajectory correct?
  Do they include all possible cases?

Motivations from PDE's

In the Fourier basis PDE can be seen as infinitely many ODE's

$$\dot{a}_k(t) = F(a_0, a_1, a_{-1}, a_2, a_{-2}, \dots).$$

We decompose variables as

$$\begin{aligned} x(t) &= (a_0(t), a_1(t), a_{-1}(t), \dots, a_N(t), a_{-N}(t)) \\ y(t) &= (a_{N+1}(t), ...) \end{aligned}$$

So our system of equation becomes

$$\begin{aligned} x'(t) &= f(x(t), y(t)) \\ y'(t) &= g(x(t), y(t)) \end{aligned}$$

When we have some a priori bounds for $y(t)$ then it is enough to integrate (3) and treat $y(t)$ as a perturbation.

Direct use of ODE integrators

Perturbed oscylator

$$
\begin{aligned}
x' &\in y + [-\epsilon, \epsilon], \\
y' &\in -x + [-\epsilon, \epsilon].
\end{aligned}
$$

Having

- ODE integrator
- interval arithmetics

one can expect that using it both he should obtain estimates for all solutions of the above equations.

< Show program. >

Direct use of ODE integrators

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$$
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x' &\in y + [-\epsilon, \epsilon], \\
y' &\in -x + [-\epsilon, \epsilon].
\end{aligned}
$$

Fixed parameters

For fixed $\delta \in [-\epsilon, \epsilon]^2$

$$
\begin{aligned}
x' &= y + \delta_1, \\
y' &= -x + \delta_2,
\end{aligned} \tag{3}
$$

All solutions are BOUNDED!

We have the energy integral $(x - \delta_2)^2 + (y + \delta_1)^2$.

Direct use of ODE integrators

Perturbed oscylator

$$\begin{aligned} x' &\in y + [-\epsilon, \epsilon], \\ y' &\in -x + [-\epsilon, \epsilon]. \end{aligned}$$

Resonant forcing

$$\begin{aligned} x' &= y, \\ y' &= -x + \epsilon \sin t. \end{aligned}$$

Solution is UNBOUNDED!

Algorithm

The goal

For given:

- equation

$$x'(t) = f(x(t), y(t)), \quad x(0) = x_0$$

  where
  - $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is C^1
  - $y : \mathbb{R} \supset D \to \mathbb{R}^m$
    We assume only that y is bounded and measurable
- starting set $[x_0] \subset \mathbb{R}^n$,
- set of admissible perturbations (controls) $[y_0]$

Find rigorous bounds for $\varphi(t, [x_0], [y_0])$.

Algorithm

One step of the algorithm

Input data:

- t_k is a current time
- h_k is a time step
- $[x_k] \subset \mathbb{R}^n$, such that $\varphi(t_k, [x_0], [y_0]) \subset [x_k]$

Output data:

- $t_{k+1} = t_k + h_k$ is a new current time
- $[x_{k+1}] \subset \mathbb{R}^n$, such that $\varphi(t_{k+1}, [x_0], [y_0]) \subset [x_{k+1}]$

Algorithm

1. **Generation of a rough enclosure for φ**
   We find interval sets $[W_2] \subset \mathbb{R}^n$ and $[W_y] \subset \mathbb{R}^m$, such that

$$\varphi([0,h_k],[x_k],[y_k]) \subset [W_2]$$
$$[y_k]([0,h_k]) \subset [W_y]$$

2. We fix $y_c \in [W_y]$.
3. **Computation of an unperturbed solution.**
   From C^0-Lohner algorithm we obtain $[\overline{x}_{k+1}] \subset \mathbb{R}^n$ $[W_1] \subset \mathbb{R}^n$, such that

$$\begin{aligned} \overline{\varphi}(h_k,[x_k],y_c) &\subset [\overline{x}_{k+1}] \\ \overline{\varphi}([0,h_k],[x_k],y_c) &\subset [W_1] \end{aligned}$$

4. **Computation of the influence of the perturbation.**
   We find a set $[\Delta] \subset \mathbb{R}^n$, such that

$$\varphi(t_{k+1},[x_0],[y_0]) \subset [x_{k+1}] = [\overline{x}_{k+1}] + [\Delta].$$

Computation of the influence of the perturbation.

Let

- x_1 be a weak solution of the unperturbed system

$$x_1' = f(x_1, y_c), \quad x_1(t_0) = x_0,$$

- x_2 be a weak solution of the perturbed system

$$x_2' = f(x_2, y(t)), \quad x_2(t_0) = \bar{x}_0.$$

- $x_1(t) \in [W_1], \quad x_2(t) \in [W_2], \qquad \text{for } t \in [t_0, t_0 + h].$

An approach based on logarithmic norms

Logarithmic norm of matrix A

$$\mu(A) = \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h}$$

For any $t \in [0, h]$ holds

$$\|x_2(t_0 + t) - x_1(t_0 + t)\| \leq$$
$$e^{lt}\|x_1(t_0) - x_2(t_0)\| + e^{lt} \int_{t_0}^{t_0+t} e^{-ls}\|[\delta]\|ds =$$
$$e^{lt}\|x_1(t_0) - x_2(t_0)\| + \frac{\|[\delta]\|}{l}(e^{lt} - 1)$$

where

$$l = \sup\left(\mu\left(\frac{\partial f}{\partial x}([W_2], y_c)\right)\right)$$

$$[\delta] = \{f(x, y_c) - f(x, y) \mid x \in [W_2], y \in [W_y]\}.$$

A component-wise estimates

The following inequality holds for $t \in [t_0, t_0 + h]$ and $i = 1, \ldots, n$

$$|x_{1,i}(t) - x_{2,i}(t)| \leq \left(e^{Jt} \cdot (x_0 - \bar{x}_0)\right)_i + \left(\int_{t_0}^{t} e^{J(t-s)} C \, ds\right)_i , \qquad (3)$$

where

$$\begin{array}{rcl} [\delta] & = & \{f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y]\}, \\ C_i & \geq & \sup |[\delta_i]| , \quad i = 1, \ldots, n \\ J_{ij} & \geq & \begin{cases} \sup \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) & \text{if } i = j, \\ \sup \left| \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) \right| & \text{if } i \neq j. \end{cases} \end{array}$$

< Run program>

Component-wise estimates vs Logarithmic norms

- In general compotent-wise estimates (CW method) should be better than estimates based on logarithmic norms (LN method).
- This especially can be seen in the systems where on one coordinate there is strong stretching while on the another there is strong contraction.
- LN method can be better in some special situations.
- These two methods can be easilly combined. We can divide coordinates into groups and for some of them use CW method while for the rest of them use LN method.

CW method vs LN method

time step h	LN method D	CW method D_1, D_2
0.799	0.112996	0.122332
0.7	0.0989949	0.101375
0.66	0.0933381	0.0934792
0.658	0.0930553	0.0930927
0.657	0.0929138	0.0928997
0.65	0.0919239	0.0915541
0.5	0.0707107	0.0648721
0.25	0.0353553	0.0284025
0.1	0.0141421	0.0105171
0.01	0.00141421	0.00100502
0.001	0.000141421	0.00010005

Table: Perturbed harmonic oscillator $\epsilon_1 = \epsilon_2 = 0.1$: Estimates of perturbations for various time steps - comparison between LN and CW method

CW method vs LN method

time step	LN method	CW method	
h	D	D_1	D_2
0.8	0.08	0.0337435	0.0888106
0.5	0.05	0.0127626	0.0521095
0.25	0.025	0.0031413	0.0252612
0.1	0.01	0.0005004	0.0100167
0.01	0.001	5.0e-06	0.0010001
0.001	0.0001	5.002e-08	0.0001

Table: Perturbed harmonic oscillator $\epsilon_1 = 0, \epsilon_2 = 0.1$: Estimates of perturbations for various time steps - comparison between LN and CW method

Conclusions

Presented algorithm was used

- estimate influence of perturbation for some ODEs,
- compute poincare maps of some differential inclusions,
- prove the existence of multiple periodic orbits in the Kuramoto-Sivashinsky PDE,
- integrate delay differential equations.

Thank you for your attention!