About the Fate of the Rotating Eight Families

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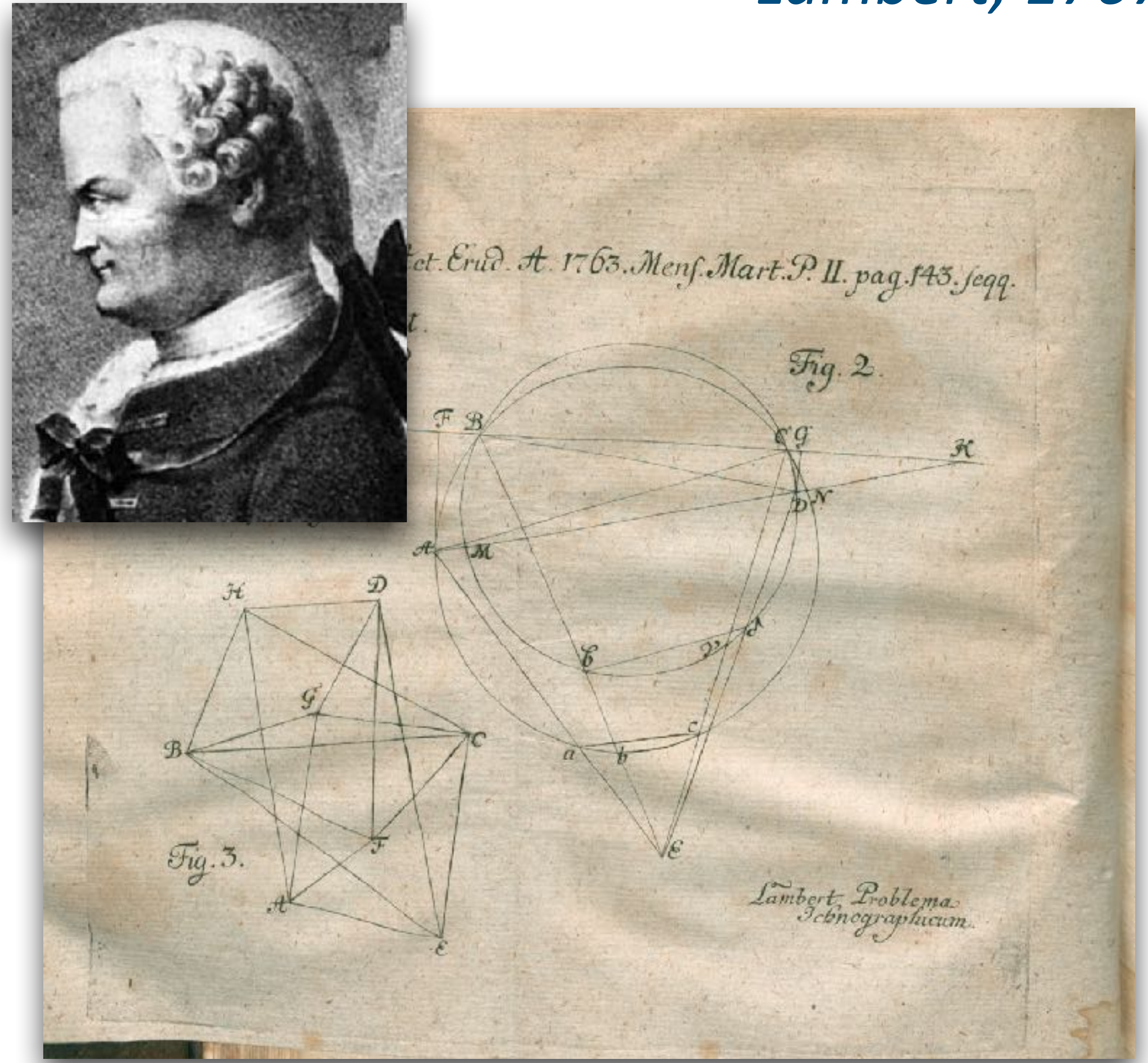
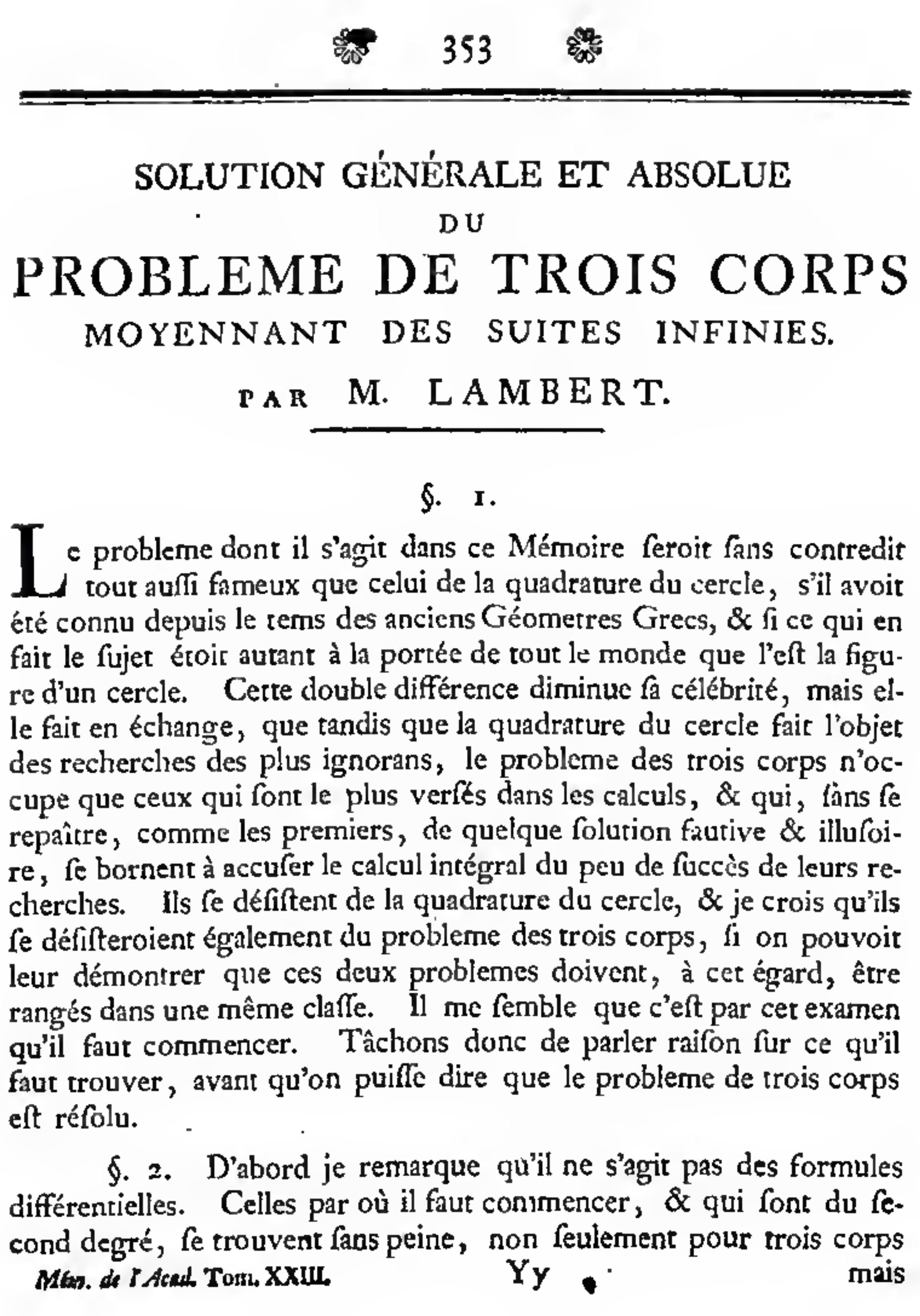
A problem is solved if its numerical integration allows to obtain the solution with precision

Lambert, 1767





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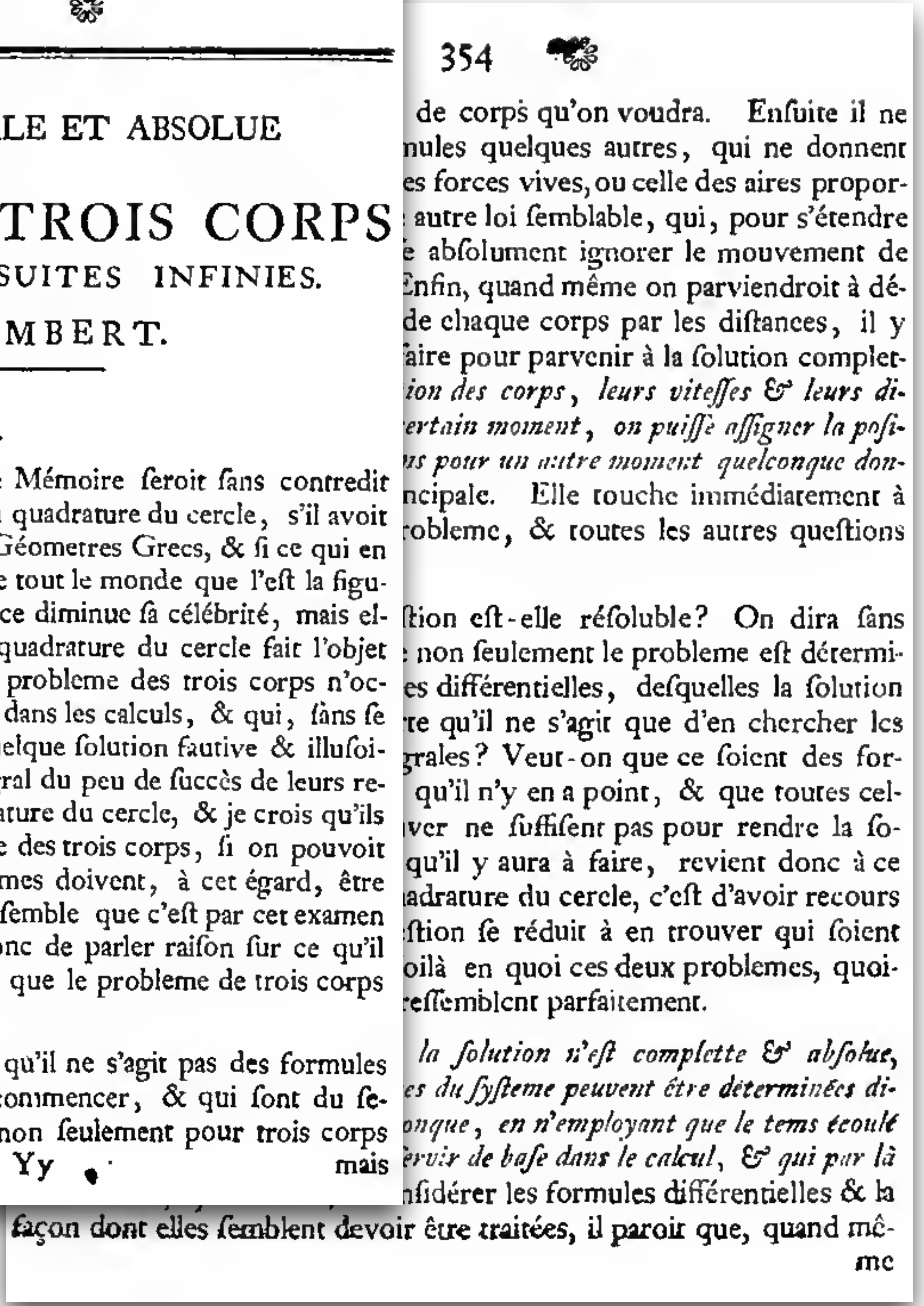
SOLUTION GÉNÉRALE ET ABSOLUE
DU
PROBLEME DE TROIS CORPS
MOYENNANT DES SUITES INFINIES.
PAR M. LAMBERT.

§. 1.

Le probleme dont il s'agit dans ce Mémoire feroit ſans contredit tout auſſi fameux que celui de la quadrature du cercle, s'il avoit été connu depuis le tems des anciens Géometres Grecs, & ſi ce qui en fait le ſujet étoit autant à la portée de tout le monde que l'eſt la figure d'un cercle. Cette double différence diminue ſa célébrité, mais elle le fait en échange, que tandis que la quadrature du cercle fait l'objet des recherches des plus ignorans, le probleme des trois corps n'occupe que ceux qui ſont le plus verſés dans les calculs, & qui, ſans ſe repaître, comme les premiers, de quelque ſolution fautive & illuſoire, ſe bornent à accuſer le calcul intégral du peu de ſuccès de leurs recherches. Ils ſe déſiſtent de la quadrature du cercle, & je crois qu'ils ſe déſiſteroient également du probleme des trois corps, ſi on pouvoit leur démontrer que ces deux problemes doivent, à cet égard, être rangés dans une même claſſe. Il me ſemble que c'eſt par cet examen qu'il faut commencer. Tâchons donc de parler raiſon ſur ce qu'il faut trouver, avant qu'on puiſſe dire que le probleme de trois corps eſt réſolu.

§. 2. D'abord je remarque qu'il ne s'agit pas des formules différentielles. Celles par où il faut commencer, & qui ſont du ſecond degré, ſe trouvent ſans peine, non ſeulement pour trois corps mais

Mém. de l'Acad. Tom. XXIII. Yy



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de corps qu'on voudra. Enſuite il ne
nules quelques autres, qui ne donnent
es forces vives, ou celle des aires propor-
autre loi ſemblable, qui, pour s'étendre
e abſolument ignorer le mouvement de
Enfin, quand même on parviendroit à dé-
de chaque corps par les diſtances, il y
aire pour parvenir à la ſolution complet-
ion des corps, leurs viteſſes & leurs di-
ertain moment, on puiſſe aſſigner la poſi-
us pour un autre moment quelconque don-
ncipale. Elle touche immédiatement à
robleme, & toutes les autres queſtions

ſtion eſt-elle réſoluble? On dira ſans
non ſeulement le probleme eſt détermi-
es différentielles, deſquelles la ſolution
re qu'il ne s'agit que d'en chercher les
grales? Veut-on que ce ſoient des for-
qu'il n'y en a point, & que toutes cel-
ver ne ſuffiſent pas pour rendre la ſo-
qu'il y aura à faire, revient donc à ce
adrature du cercle, c'eſt d'avoir recours
ſtion ſe réduit à en trouver qui ſoient
oilà en quoi ces deux problemes, quoi-
reſſemblent parfaitement.

la ſolution n'eſt complette & abſolue,
es du ſyſteme peuvent être déterminées di-
onque, en n'employant que le tems écoulé
ervir de baſe dans le calcul, & qui par là
nſidérer les formules différentielles & la
façon dont elles ſemblent devoir être traitées, il paroit que, quand même

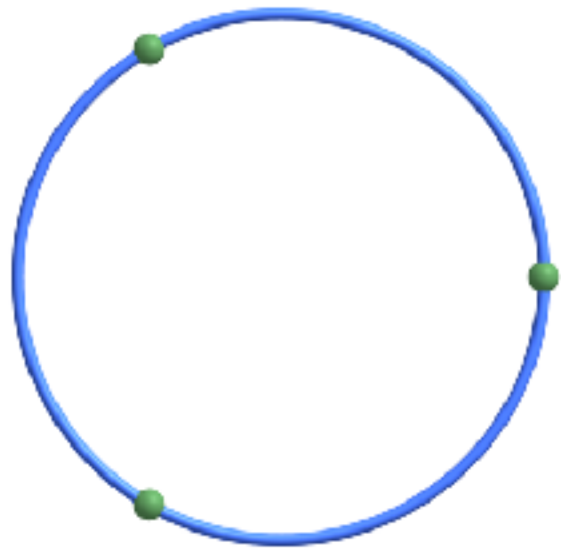
Three-body problem, the Eight and choreographies

Equations of motion $m_i \ddot{q}_i = -\nabla_i U(q), \qquad U(q) = -\sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|}$

18th-century

Euler (collinear configuration) 1767

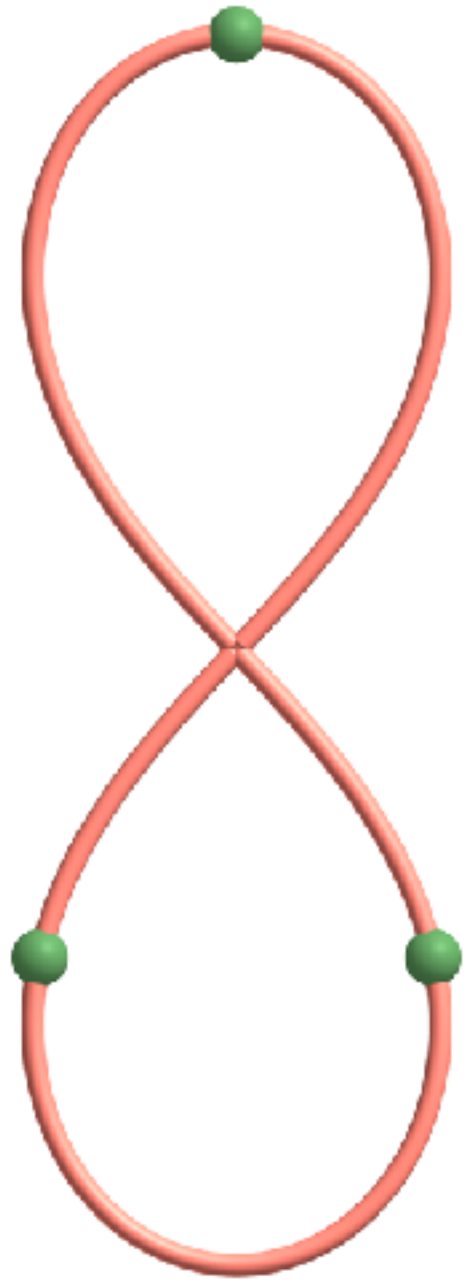
Lagrange (equilateral triangle) 1772



… skipping 200+ years of celestial mechanics…

1993: Moore discovers numerically the **Eight** (equal mass)

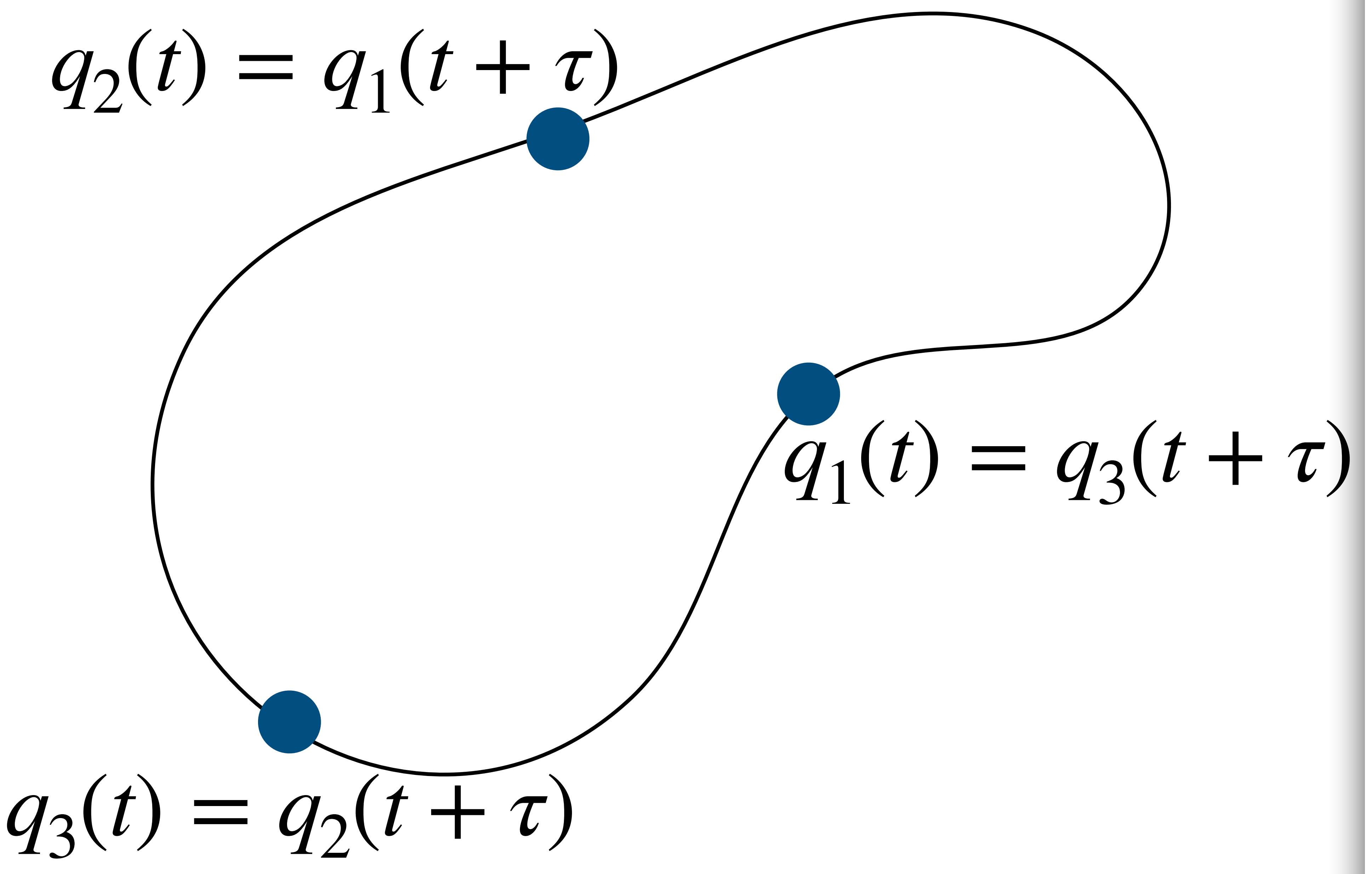
2000: Chenciner & Montgomery prove its existence

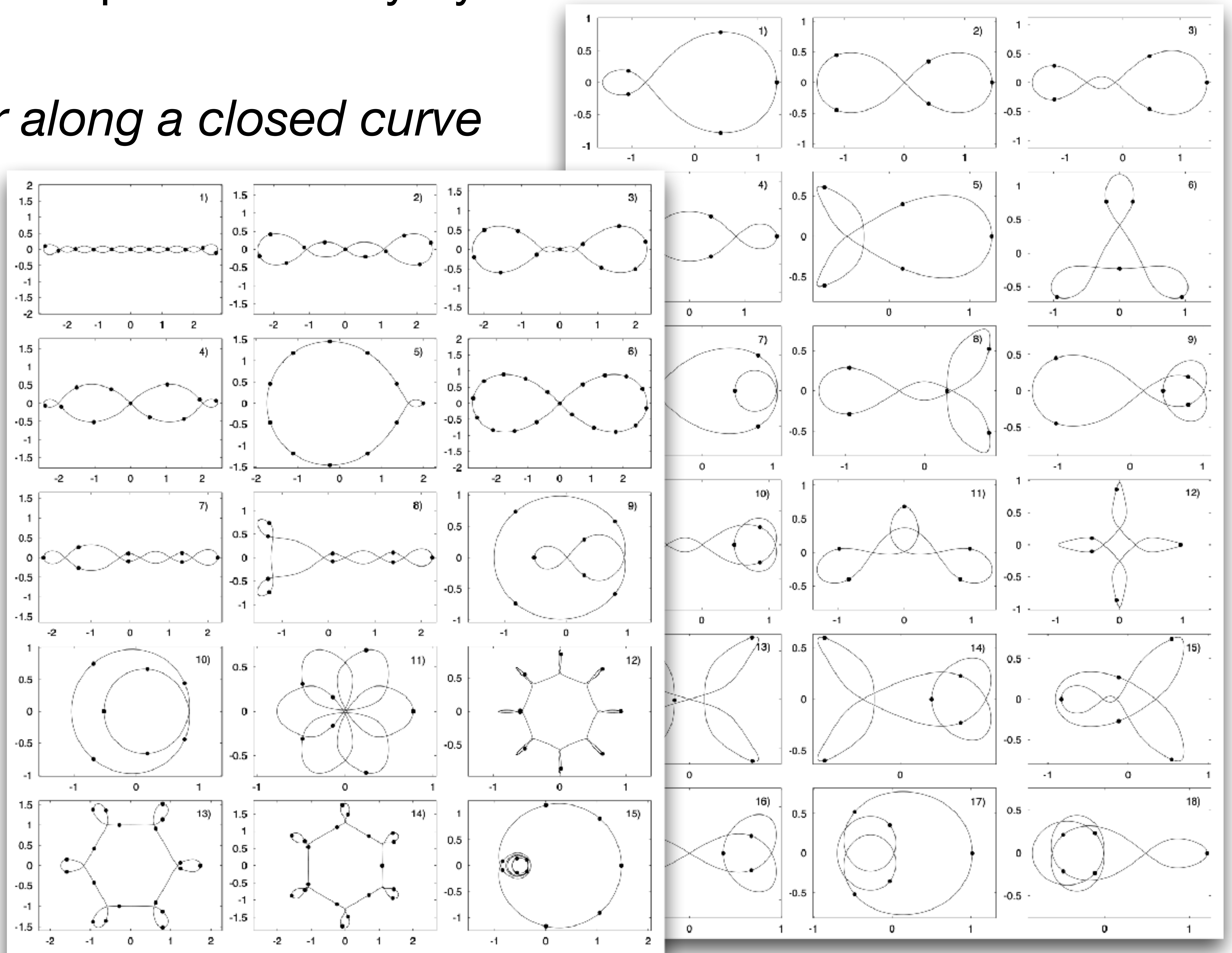


Three-body problem, the Eight and choreographies

2000s: a zoo of <u>choreographies</u> computed notably by Carles Simó

Three bodies chasing each other along a closed curve





New Families of Solutions in N-body Problems, 2001

The three Γ_i families

Setting Rotation Ω (parameter) of the inertial frame around a fixed axis
Families of (relative) choreographies bifurcating from the Eight
(in the yOz) plane in inertial frame ($\Omega = 0$)

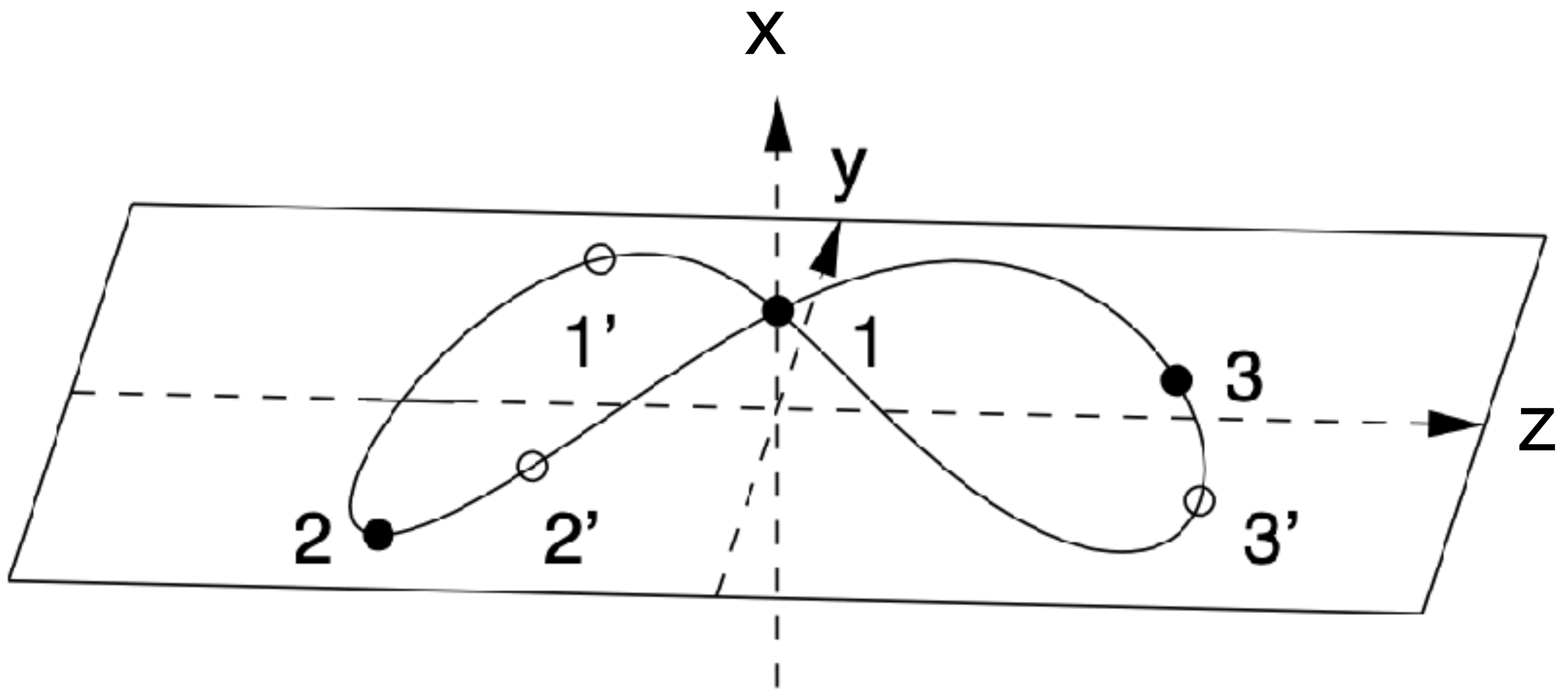
Theorem (Chenciner, Féjoz, Montgomery, 2005)

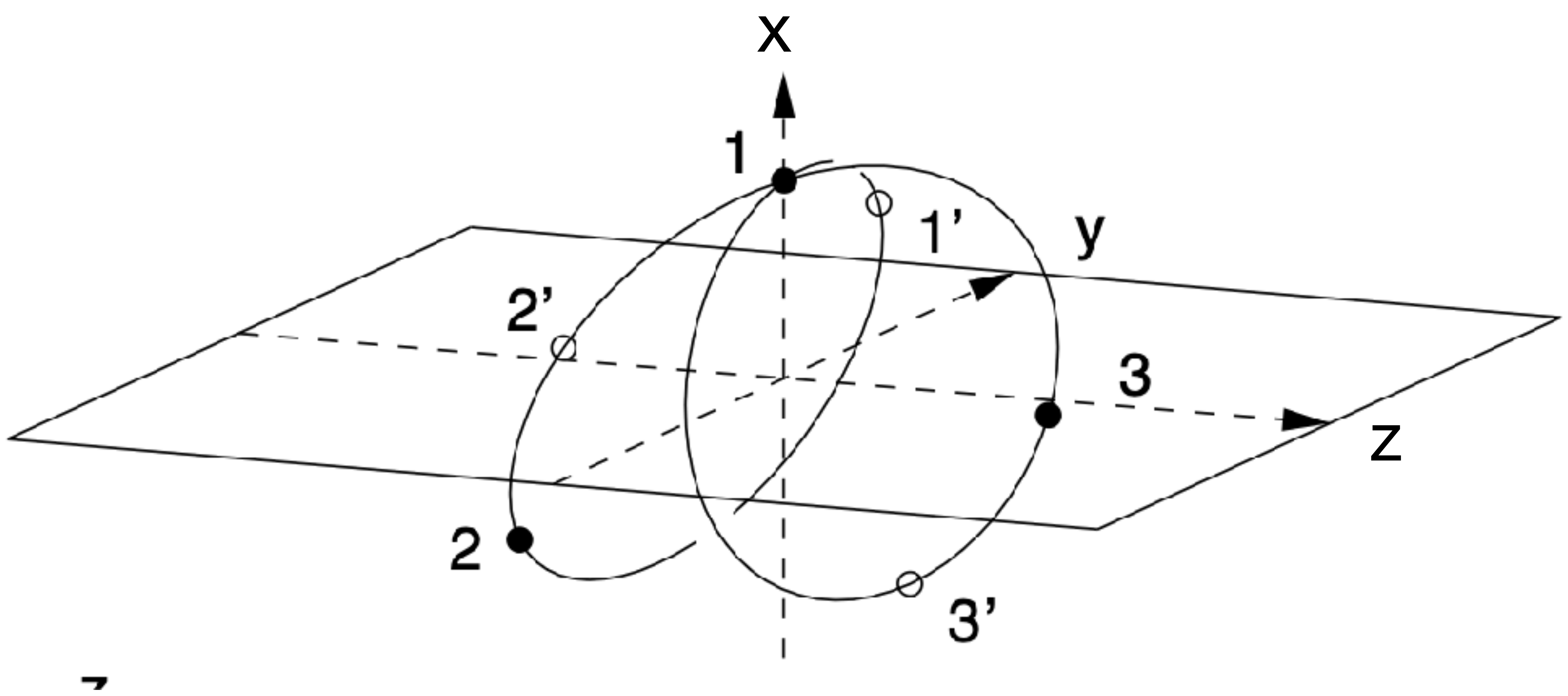
Three families $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ bifurcate from the Eight at $\Omega = 0$, one per axis

Each family has a symmetry group isomorphic to D_6

The three Γ_i families

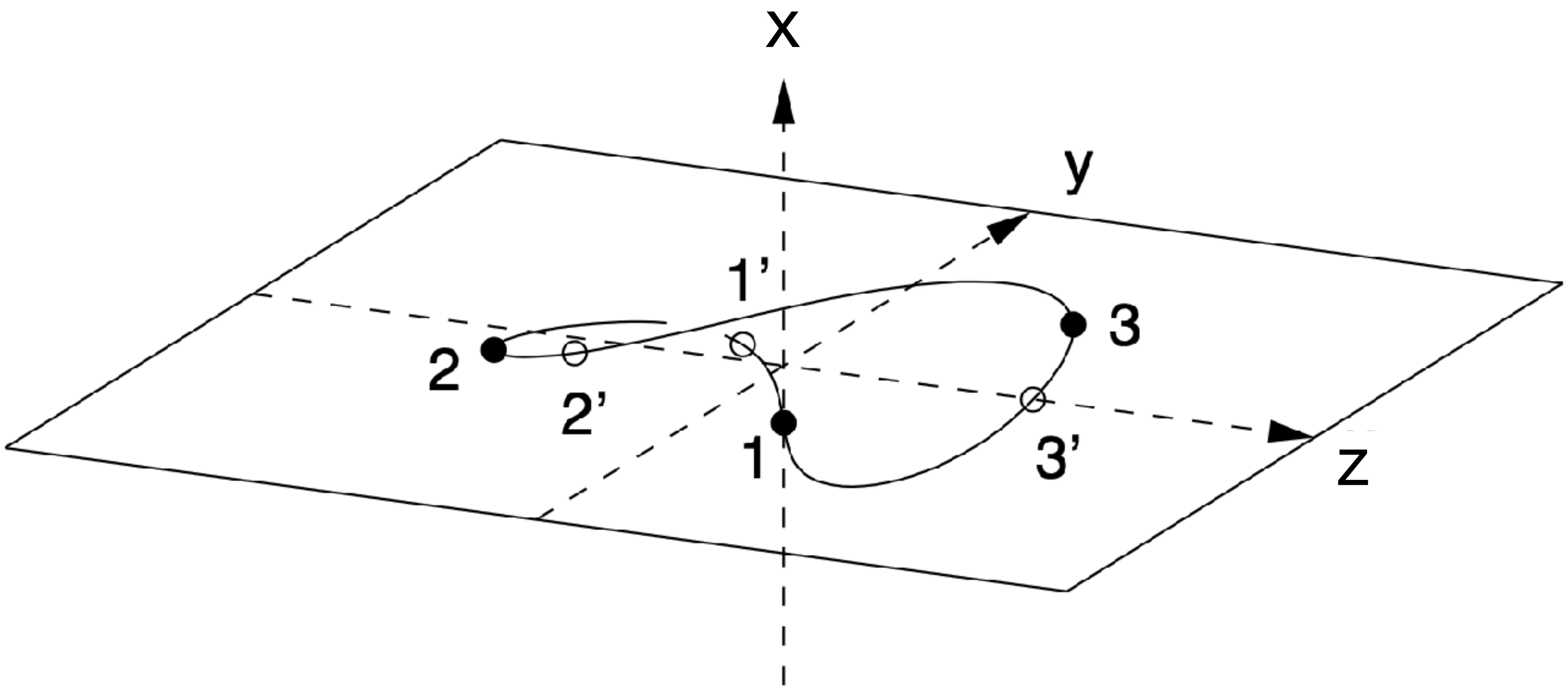
Γ_1 rotation around Oz





The three Γ_i families

Γ_2 rotation around Oy



The three Γ_i families

Γ_3 rotation around Ox

Today’s question

What is the fate of the three Γ_i branches bifurcating from the Eight?

Global continuation problem, computational in nature

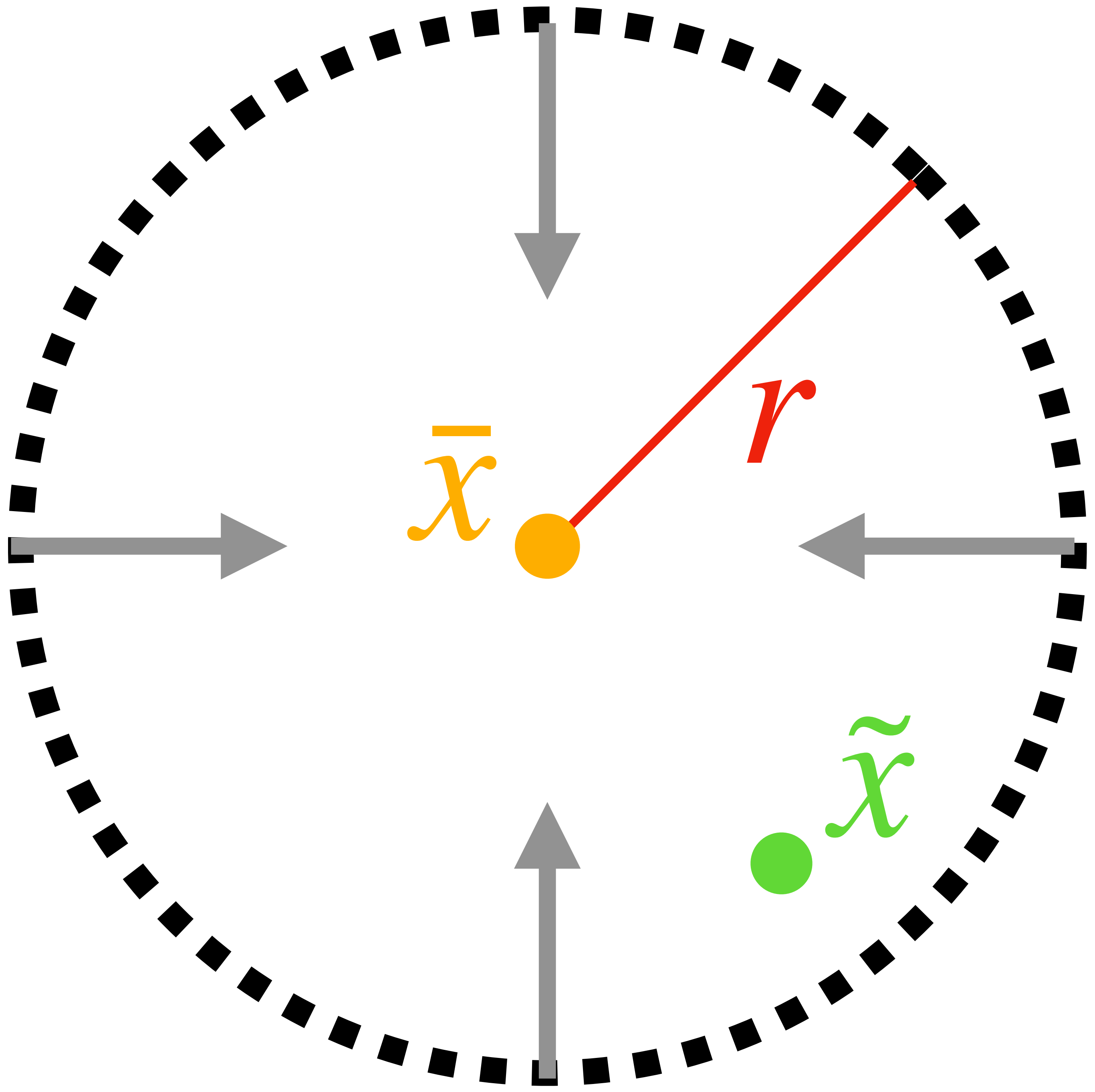
Roadmap

- A posteriori validation method
- $\Gamma_1 = P_{12}$ (proof of Marchal’s conjecture)
- Γ_2 and its relationship with Γ_1
- Comments and Γ_3

I. A posteriori validation methods

Setup

- $F : X \to X'$ between Banach spaces
- numerical approximation $\bar{x}$ of a zero of F
- Approximate inverse $A \approx DF(\bar{x})^{-1}$



Fixed points of $T(x) = x - AF(x)$ are zeros of F

Strategy

contraction on a closed ball $B(\bar{x}, r)$

Banach fixed point theorem implies the existence of an exact solution in $B(\bar{x}, r)$

I. A posteriori validation methods

Three bounds

$\|AF(\bar{x})\| \leq Y$ ⟶ accuracy of $\bar{x}$

$\|ADF(\bar{x}) - I\| \leq Z_1$ ⟶ quality of A

$\|A(DF(x) - DF(\bar{x}))\| \leq Z_2\|x - \bar{x}\|, \qquad x \in B(\bar{x}, R)$ ⟶ local control on the nonlinearities

Quantity R = maximum error threshold

Radii polynomial

$$p(r) = Y - r(1 - Z_1) + \frac{r^2}{2} Z_2(R)$$

Theorem If there exists $r \in [0,R]$ such that $\begin{cases} p(r) \leq 0 \\ Z_1 + Z_2(R)r < 1 \end{cases}$ injectivity of A

then F has a unique zero in $B(\bar{x}, r)$

Where is the computer? pure mathematical argument behind this type of CAPs

BUT we hope to work with complicated $\bar{x}$ using the computer

I. A posteriori validation methods: a toy example

Cube root of 2

Setup $0 = F(x) = x^3 - 2$ $\qquad X = X' = \mathbb{R}$

$\bar{x} \approx \sqrt[3]{2}$

$A \approx (3\bar{x})^{-1}$

Bounds $|AF(\bar{x})| = |A(\bar{x}^3 - 2)| = Y$

$|1 - ADF(\bar{x})| = |1 - A(3\bar{x}^2)| = Z_1$

Rigorous floating-point computations with interval arithmetic

$$
\begin{aligned}
|A(DF(x) - DF(\bar{x}))| &= 3|A(x-\bar{x})(x+\bar{x})| \\
&\le \underbrace{3|A|(2|\bar{x}|+R)}_{Z_2(R)}|x-\bar{x}|
\end{aligned}
$$

I. A posteriori validation methods: a toy example VERSION 2

Cube root function $\quad 0 = F(x,\lambda) = x^3 - 2 - \lambda, \qquad \lambda \in [-1,1]$

Chebyshev polynomials $\quad T_k(\cos(\theta)) = \cos(k\theta)$

Ansatz $\quad x(\lambda) = x_0 + 2\sum_{k\geq 1} x_k T_k(\lambda)$

$$X = X' = \left\{ \{x_k\} \in \mathbb{R}^{\mathbb{N}} \;:\; \|x\|_X = |x_0| + 2\sum_{k\geq 1} |x_k| < \infty \right\}$$

$$\mathcal{F}(x) = x^{\circledast 3} - 2e_0 - e_1, \qquad e_0(\lambda) = 1,\ e_1(\lambda) = \lambda$$

in seq. space $\quad e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad e_1 = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ \vdots \end{pmatrix}$ $\qquad$ product of Cheb. poly. $\circledast : X \times X \to X$

I. A posteriori validation methods: a toy example VERSION 2

Interpolation $\lambda^{(j)} = -\cos\left(j\frac{\pi}{K}\right)$ $\bar{x}^{(j)} \approx \sqrt[3]{2+\lambda^{(j)}}$

$$\bar{x}(\lambda) = \bar{x}_0 + 2\sum_{k\geq 1} \bar{x}_k T_k(\lambda)$$

For the approximate inverse A

$$[D\mathscr{F}(\bar{x})h](\lambda) = [D_xF(\bar{x}(\lambda),\lambda)]h(\lambda), \qquad D_xF(\bar{x}(\lambda),\lambda) = 3\bar{x}(\lambda)^2$$

need to find an approximation of the function $\lambda \mapsto \left(3\bar{x}(\lambda)^2\right)^{-1}$

$$A^{(j)} \approx \left(3\left(\bar{x}^{(j)}\right)^2\right)^{-1} \qquad A(s) \approx A_0 + 2\sum_{k\geq 1} A_k T_k(s)$$

Then $M_A \approx D\mathscr{F}(\bar{x})$ since

$$[M_A D\mathscr{F}(\bar{x})]h = \left(A \circledast D_xF(\bar{x}(\,\cdot\,),\,\cdot\,)\right) \circledast h \approx h$$

I. A posteriori validation methods: a toy example VERSION 2

Bounds

$$\|M_A \mathscr{F}(\bar{x})\|_X = \|A \circledast (\bar{x}^{\circledast 3} - 2e_0 - e_1)\|_X \leq Y$$

$$\|I - M_A D\mathscr{F}(\bar{x})\|_{B(X)} = \|e_0 - A \circledast \left(3\bar{x}^{\circledast 2}\right) \|_X \leq Z_1$$

$$\|M_A(D\mathscr{F}(x) - D\mathscr{F}(\bar{x}))\|_{B(X)} \leq \underbrace{3\|A\|_X(2\|\bar{x}\|_X + R)}_{Z_2(R)}\|x - \bar{x}\|_X$$

Consequences

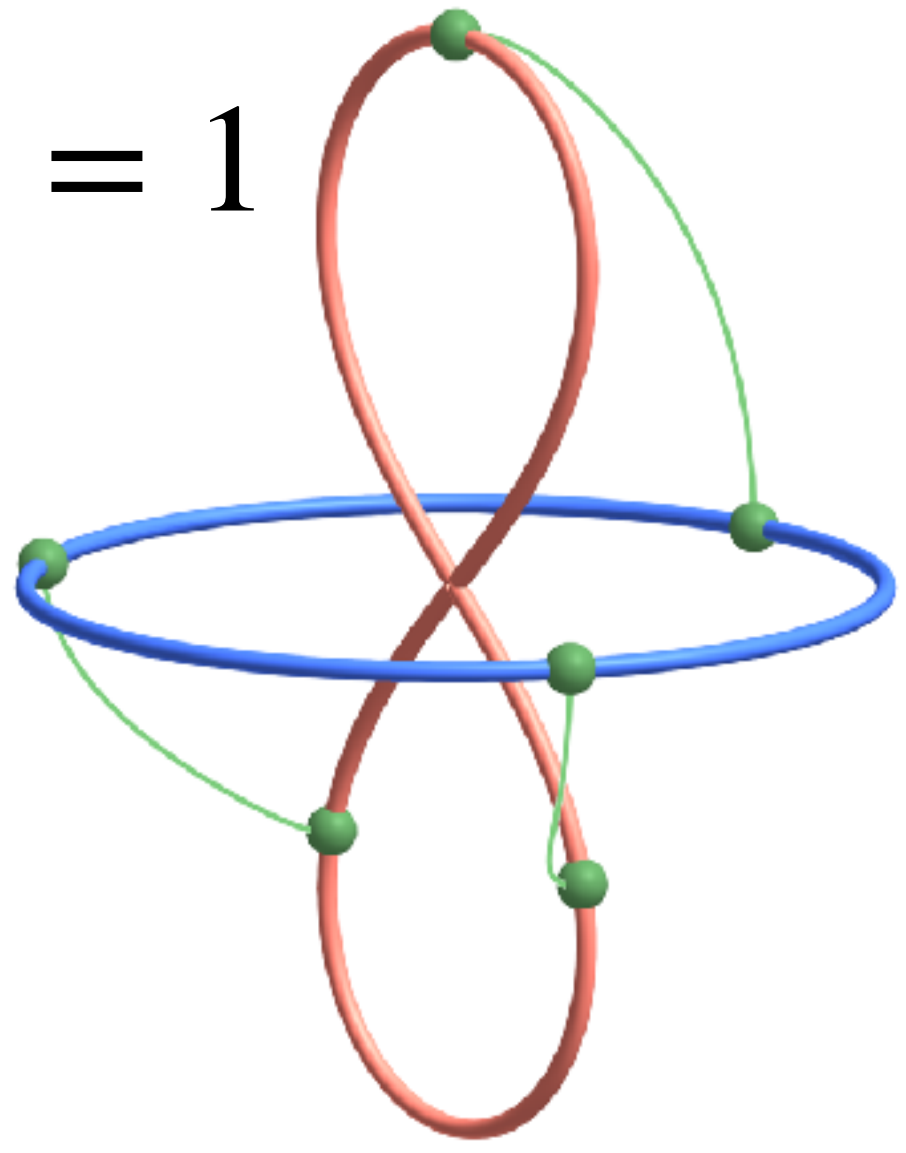
If $p(r) = Y - r\,(1 - Z_1) + \dfrac{r^2}{2} Z_2 \leq 0$ and $Z_1 + r\,Z_2 < 1$, then we have the existence of a function such that $F(\tilde{x}(\lambda), \lambda) = 0$

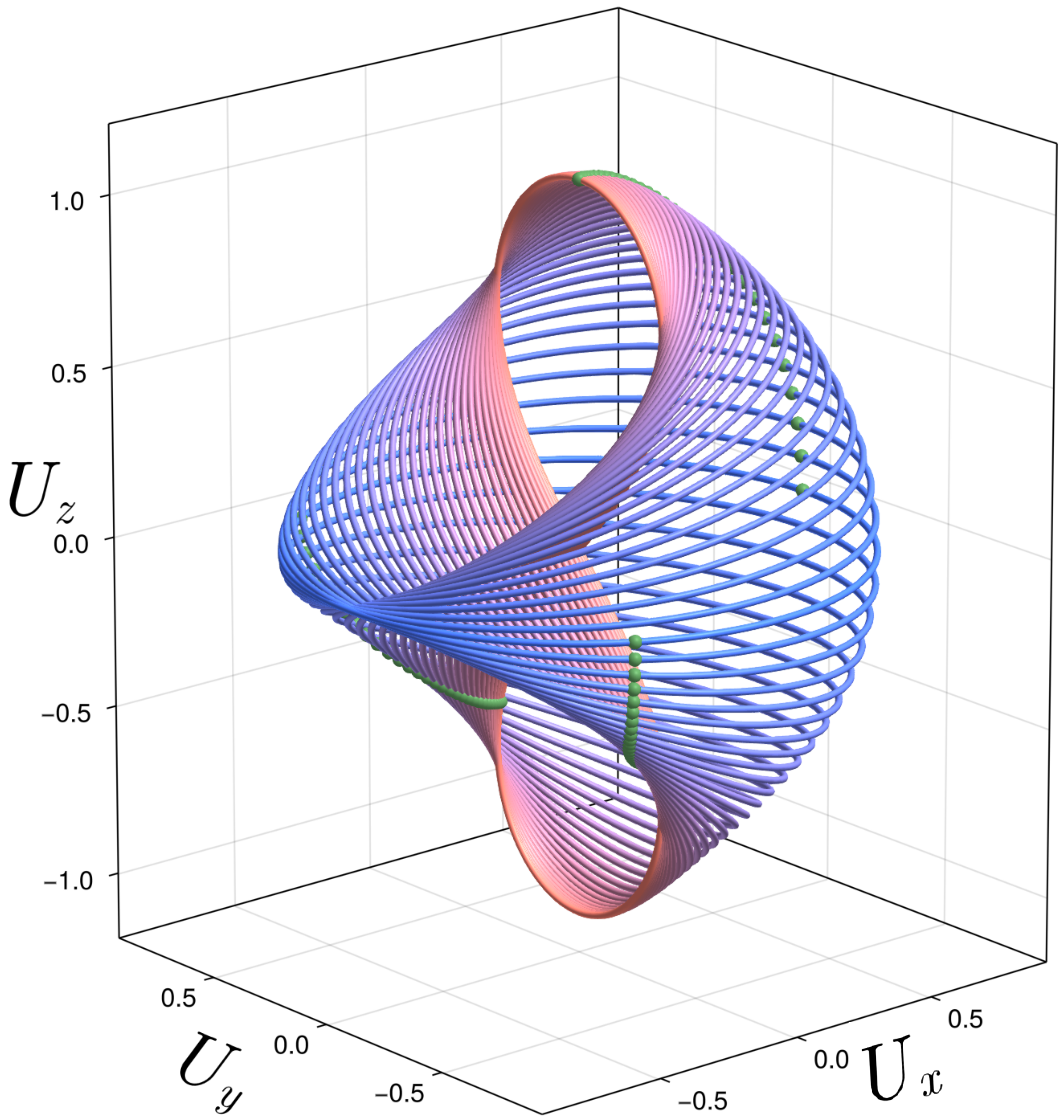
➡ no bifurcation can occur along the solution curve (IFT)

II. Fate of $\Gamma_1(= P_{12})$

Marchal (1990s) numerical continuation from the Lagrange triangle $\Omega = 1$

(travelled twice in one period)





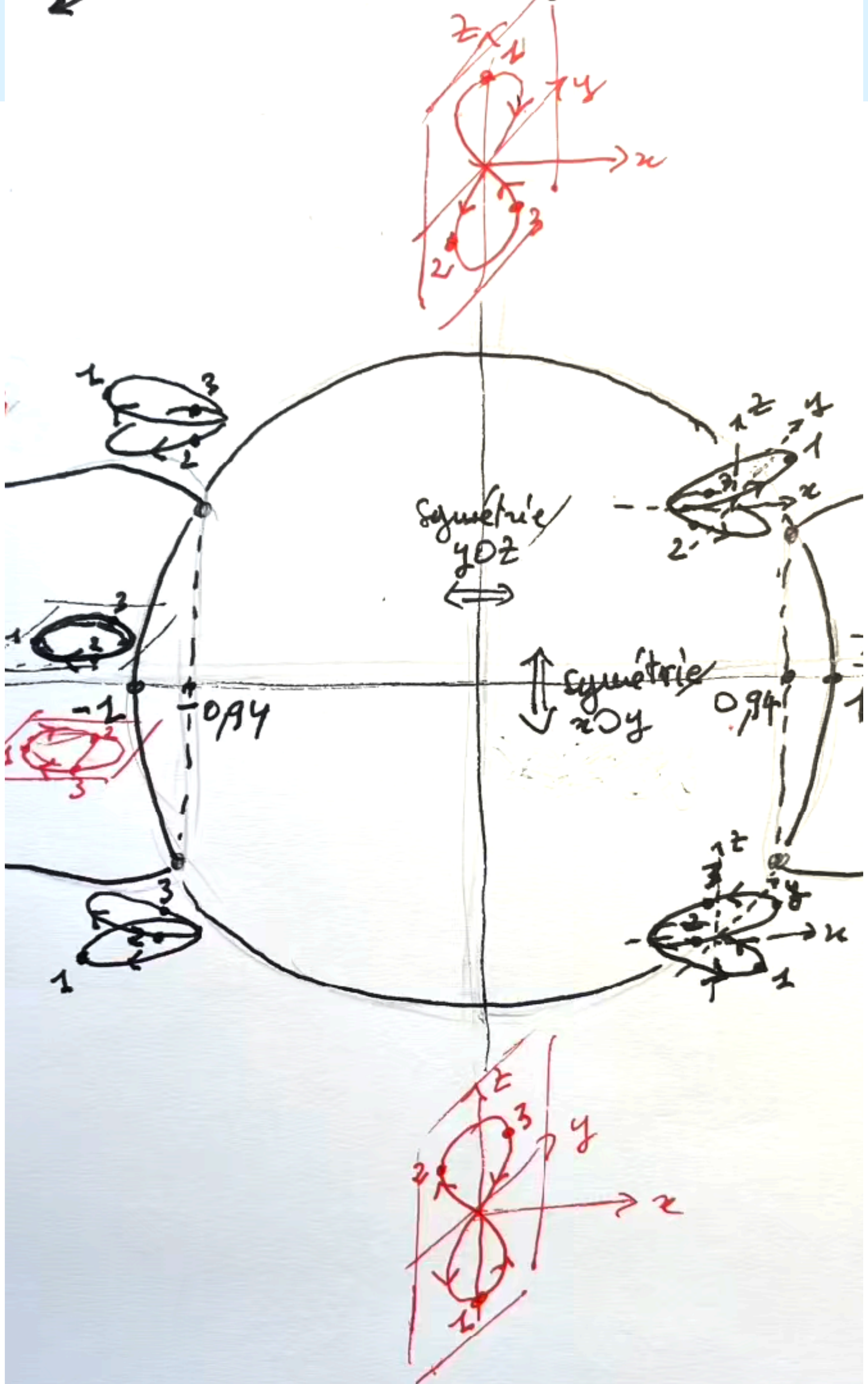
Marchal's conjecture (1999)

The Lagrange triangle and the Eight belong to the same continuation class

Theorem (Calleja, García-Azpeitia, H, Lessard, Mireles James [2026])

II. Fate of $\Gamma_1(=P_{12})$

The proven branch is 1/4 of the full Γ_1 family



II. Fate of $\Gamma_1(= P_{12})$

How the proof works…

$$U_j = e^{\hat{J}\Omega t} q_j, \qquad U_j = U_3(t + \frac{4\pi}{3} j) = S^j U, \qquad U = U_3$$

$$\ddot{U} - 2\Omega \hat{J} U - \Omega^2 \hat{I} U = \text{forces}, \qquad \hat{I} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{J} := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

II. Fate of $\Gamma_1 (= P_{12})$

Imposing the symmetries of Γ_1 does not remove all bifurcations

sing. at the Lagrange triangle (LT homographic family): blow-up $U_z = au_z$

sing. at the Eight (x-translation invariance): impose zero average in time of U_x

$$\beta e_x - \Omega^2 \bar{I} u + 2\Omega \bar{J} \dot{u} + \ddot{u} + \sum_{j=1}^{2} \frac{u - S^j u}{\|L_{\sqrt{a}} u - S^j L_{\sqrt{a}} u\|^3} = 0$$

$$u_z(0,s) - 1 = 0, \qquad \int_0^{2\pi} U_x(t,s)\, dt = 0, \qquad u = (U_y, U_y, u_z)$$

II. Fate of $\Gamma_1(=P_{12})$

By conservation of linear momentum in the x-direction, $\beta = 0$ for periodic solutions

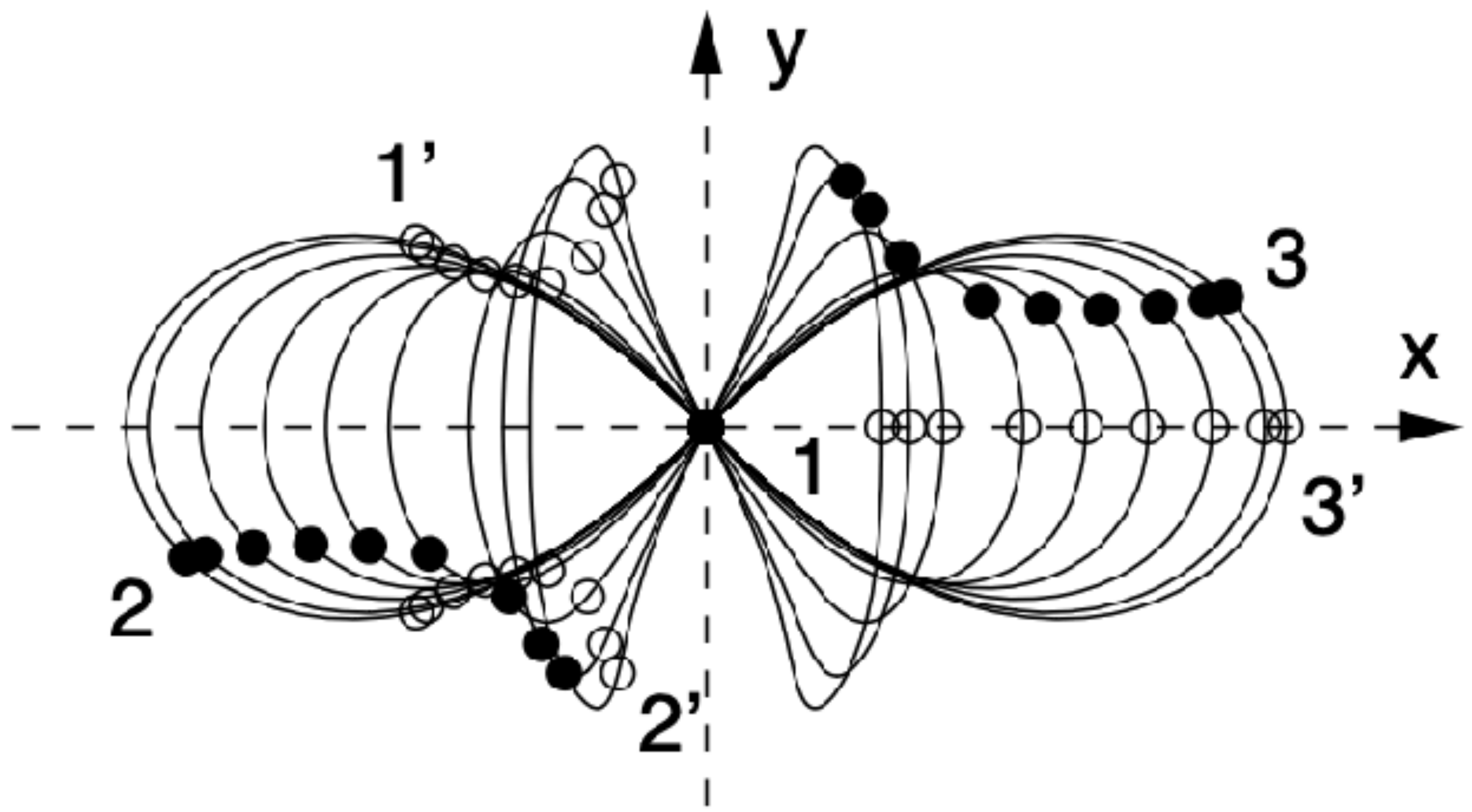
$$0 = \int_0^{2\pi} \sum_{j=1}^{3} \left\langle \ddot{U}_j + \sum_{k \neq j} \frac{U_j - U_k}{\|U_j - U_k\|^3}, e_1 \right\rangle dt$$

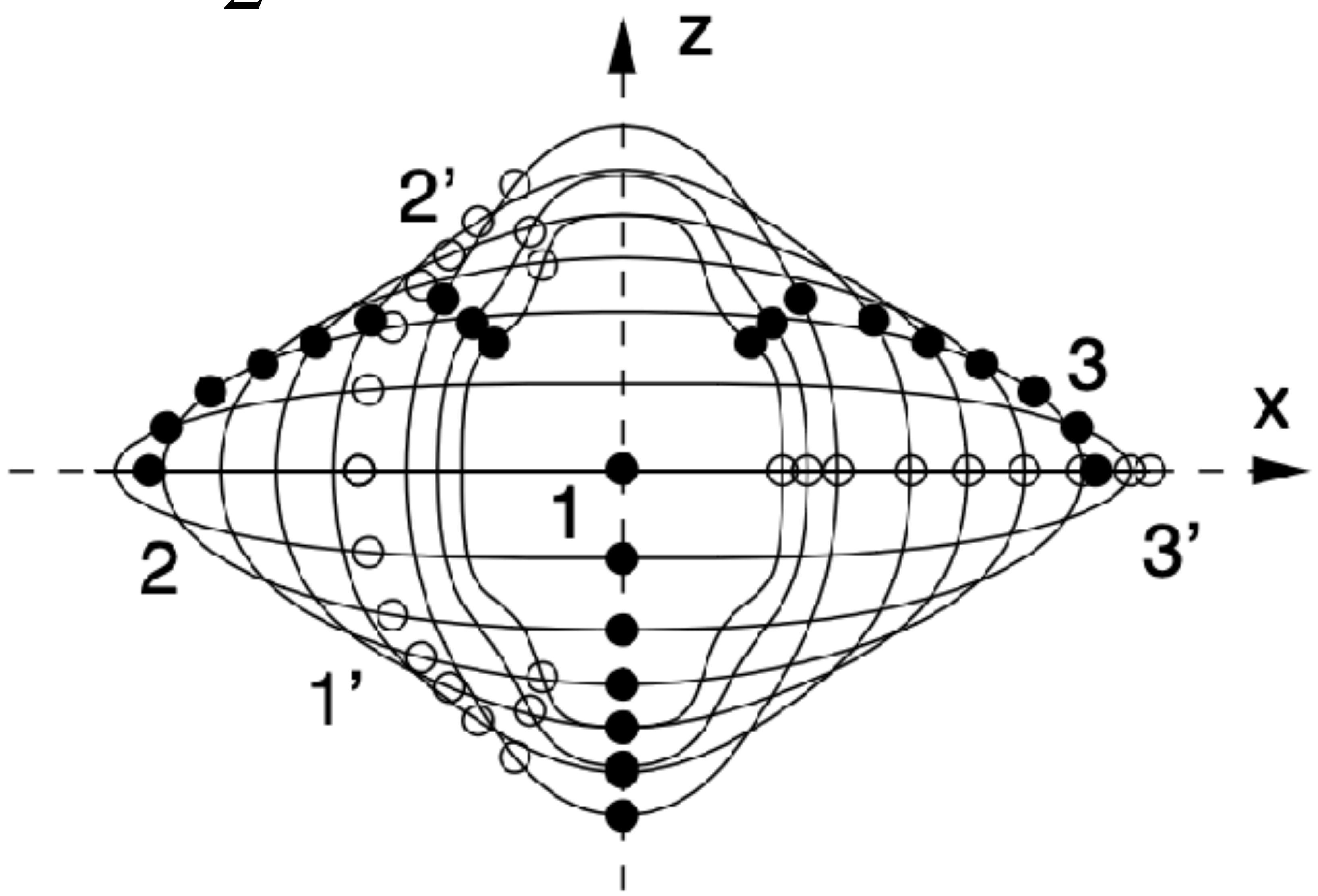
Thus (for a periodic solution u)

$$0 = \int_0^{2\pi} \sum_{j=1}^{3} S^j \left\langle \beta e_x - \Omega^2 \hat{I} U + 2\Omega \hat{J} \dot{U}, e_1 \right\rangle dt = 6\pi\beta$$

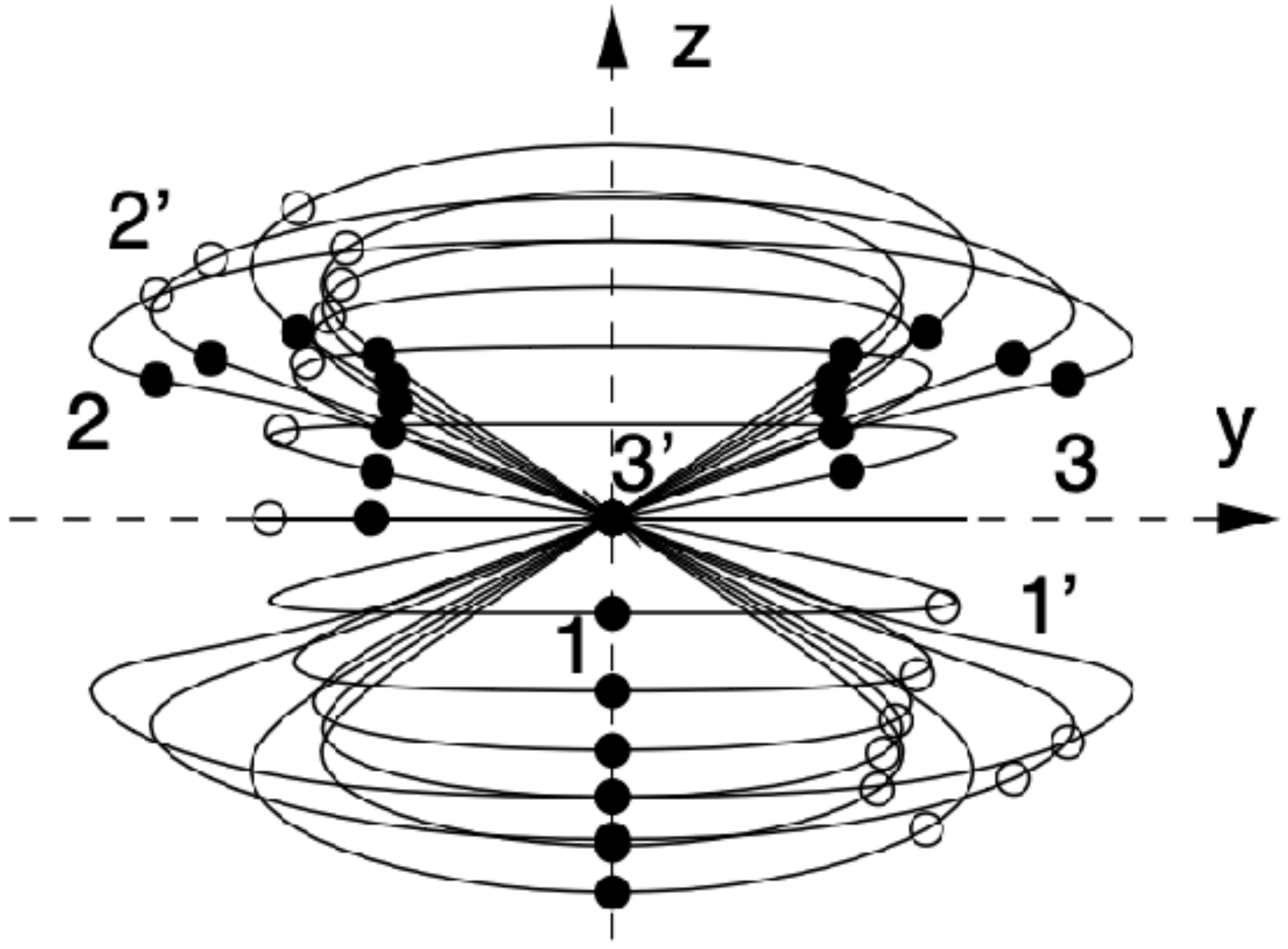
[work in progress with Renato Calleja, Alain Chenciner & Jacques Féjoz]

Chenciner, Féjoz, Montgomery (Rotating Eights I [2005]) gave Γ_2 numerically





What was missed: The branch passes through a bifurcation



III. Fate of Γ_2

Meanwhile…

Terracini and Arioli found a mountain-pass critical point of the action at $\Omega = 3/2$

They continued in Ω towards $P_{12}(=\Gamma_1)$ but did not reach the bifurcation point

What was missed: up to a rotation of the frame, this branch is (we claim) Γ_2

III. Fate of Γ_2

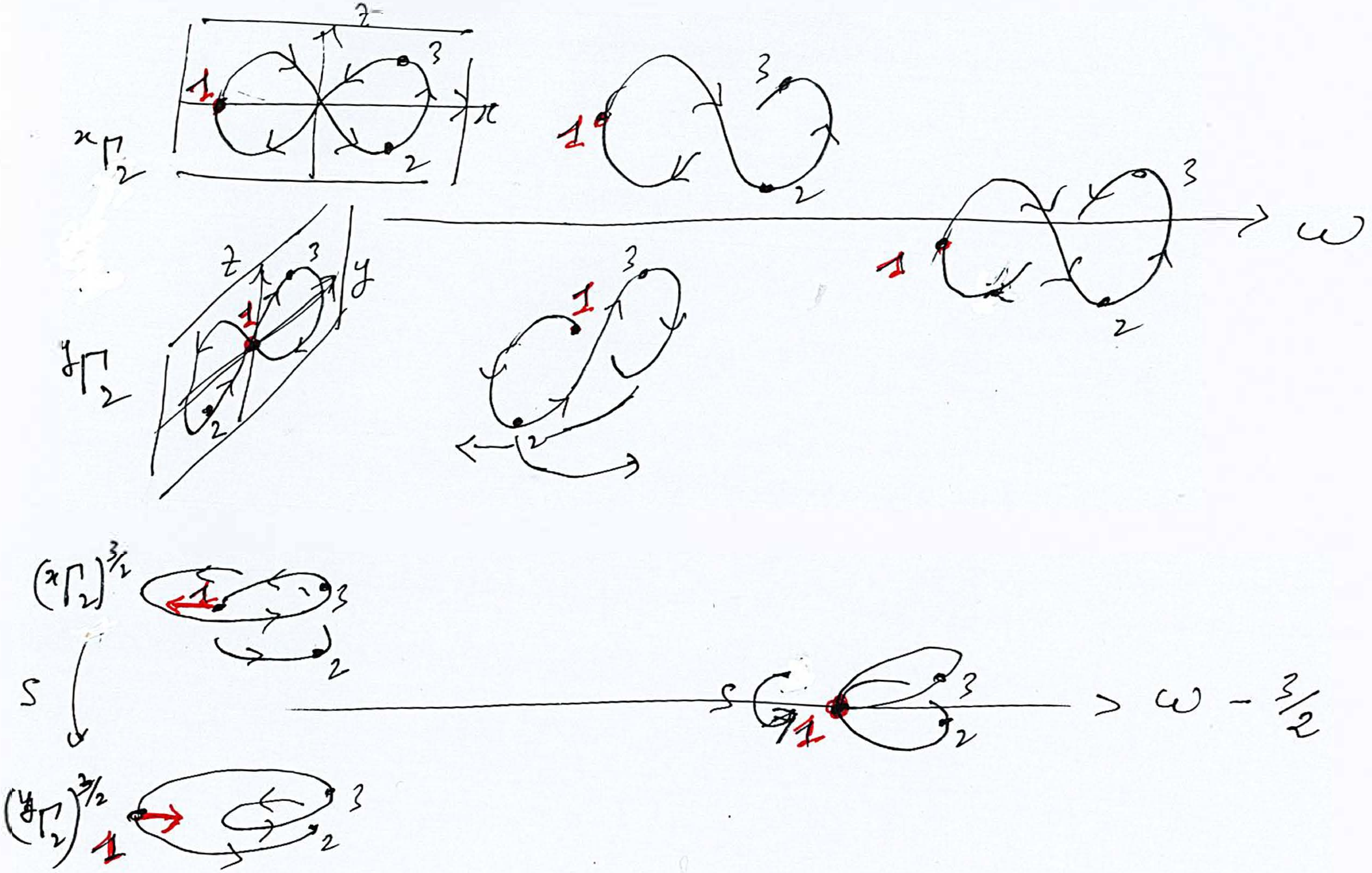
Rotating Eight version II

Γ_1 and Γ_2 are connected through

A symmetry-breaking bifurcation at some Ω_0 near 0.94, and

three and a half turns around the Oz axis during the half period $T/2$

III. Fate of Γ_2



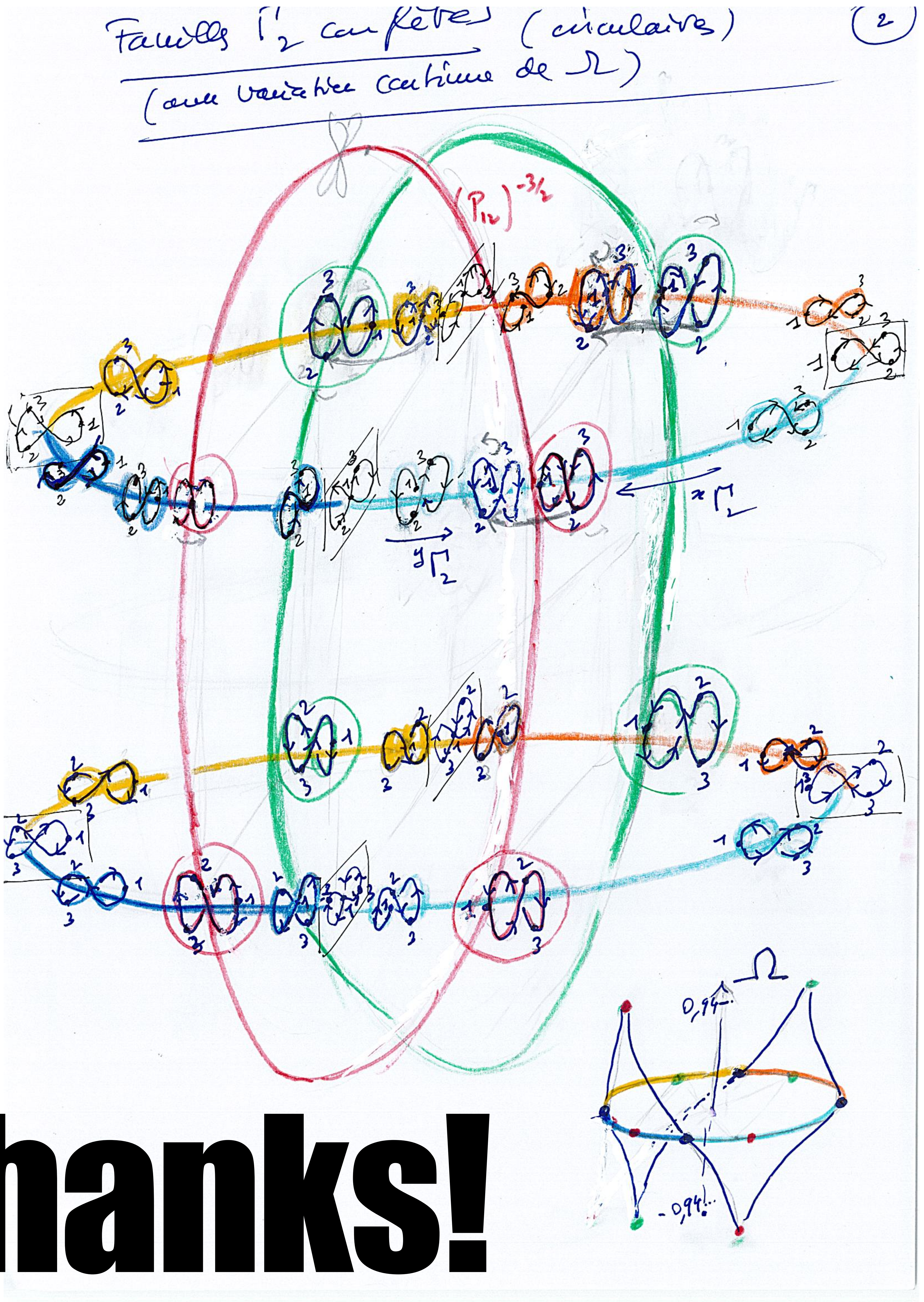
IV. Comments and Γ_3

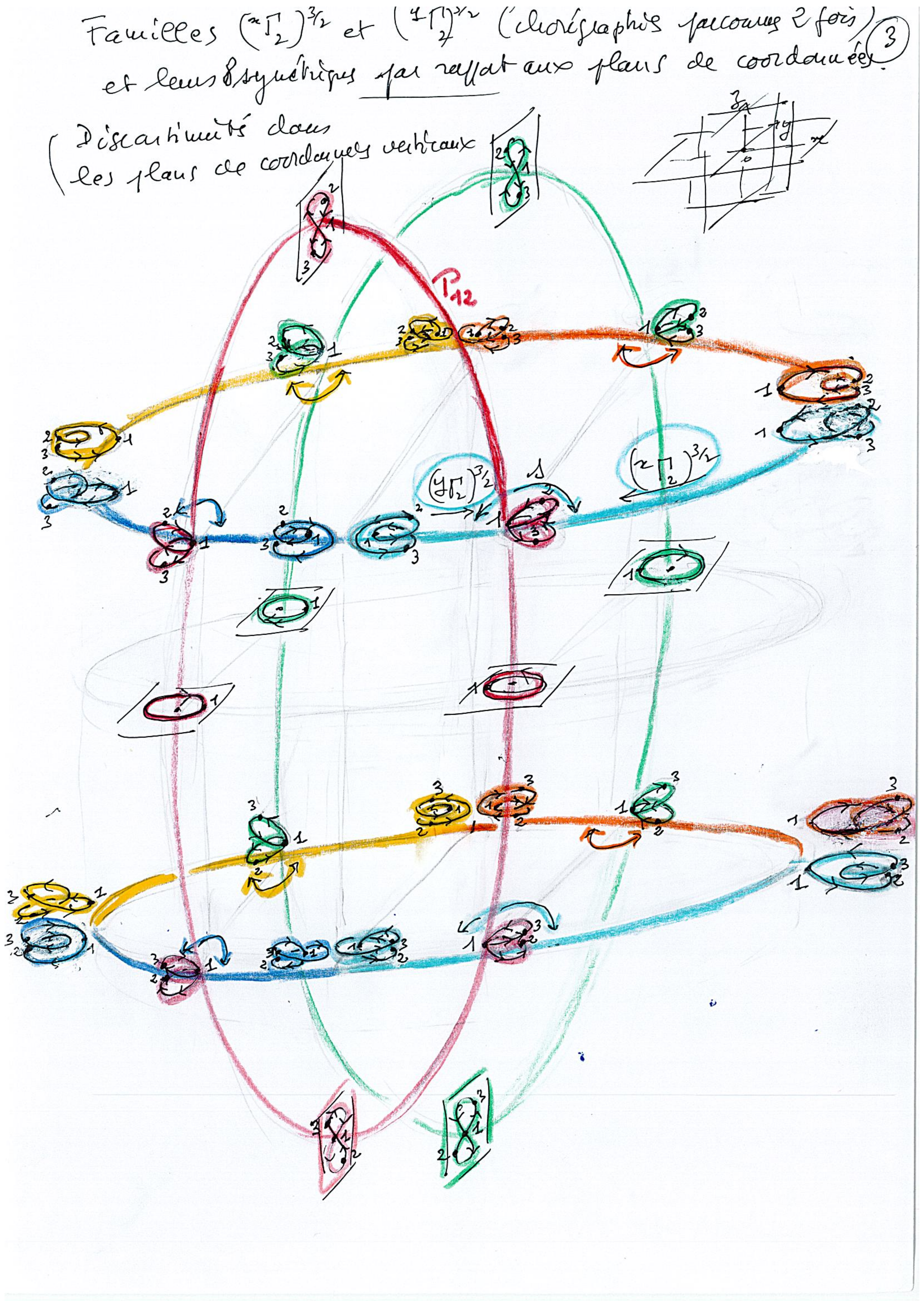
There seems to be at least one other bifurcation along Γ_1

- Where does it go?

Γ_3 (planar family) is expected to have a collision

- If two body are colliding we could regularize and see what lies beyond the collision


Familles Γ'2 complètes (circulaires)
(avec variation continue de Ω)
(P12)^-3/2
Ω
0,94
-0,94


Familles (xΓ2)^3/2 et (yΓ2)^3/2 (chorégraphies parcourues 2 fois)
et leurs 8 symétriques par rapport aux plans de coordonnées
(Discontinuités dans les plans de coordonnées verticaux
P12

Thanks!