Lower bounds on the Hausdorff dimension of some Julia sets

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Julia and Fatou sets

- Iteration of rational maps on the Riemann sphere – first studied by *Pierre Fatou* and *Gaston Julia* in the late 1910s.
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$$f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \quad \text{is a rational map,} \qquad \deg(f) \geq 2.$$

- The Julia set $J(f) \subset \widehat{\mathbb{C}}$ is the set of all Lyapunov unstable points (in the spherical metric).
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• $J(f)$ is nonemty and closed; $F(f)$ is open.
• Hausdorff dimension as a "measure" of geometric complexity of $J(f)$.

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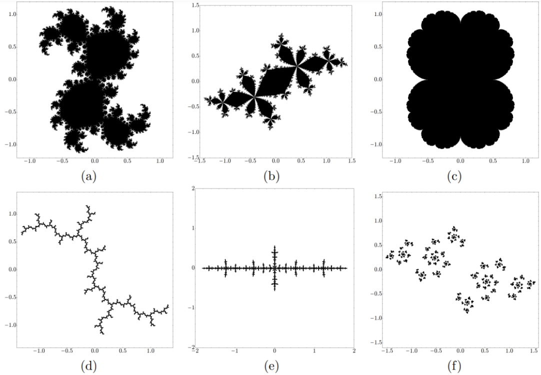
$$K_c := \{z \in \mathbb{C} \mid \limsup_{k\to+\infty} |f_c^{\circ k}(z)| < \infty\}, \qquad J_c = \partial K_c.$$



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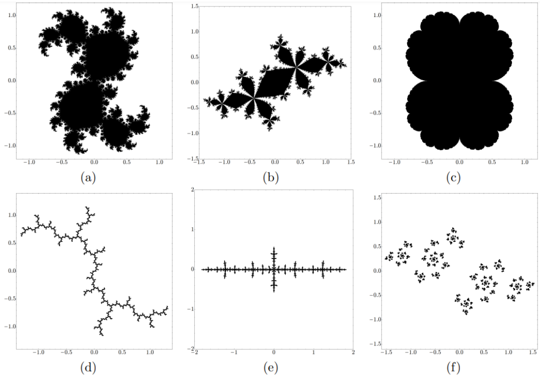
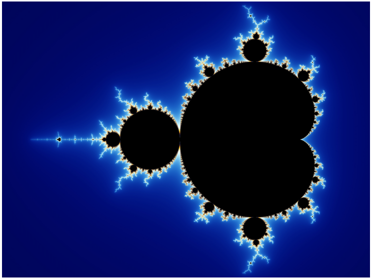


Theorem (Braverman, Yampolsky, 2006): There exist $c \in \mathbb{C}$ such that J_c is not computable.

The Mandelbrot set $\mathbb{M}$

- The Mandelbrot set: $\mathbb{M} := \{c \in \mathbb{C} \mid 0 \in K_c\}$.

$\mathbb{M}$ is the set of all parameters $c \in \mathbb{C}$, for which J_c is connected.



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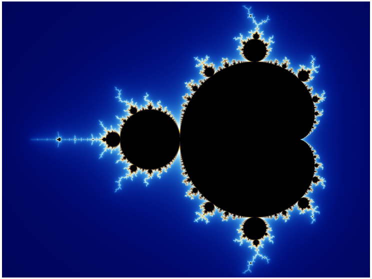

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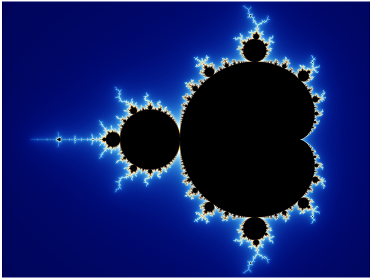

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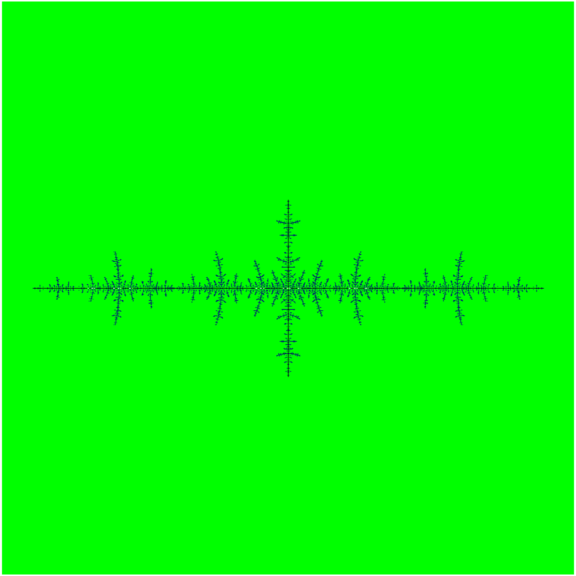
Theorem (Yoccoz, 1990s): $\mathbb{M}$ is locally connected at all non-infinitely renormalizable points.

Theorem (D. Dudko, M. Lyubich, 2023): $\mathbb{M}$ is locally connected at the period-doubling Feigenbaum parameter $c_F \approx -1.401155189$.

Zoom into J_{c_F}.



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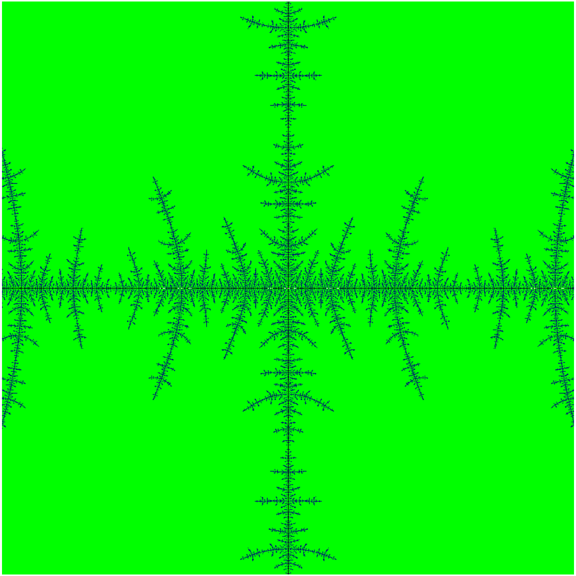

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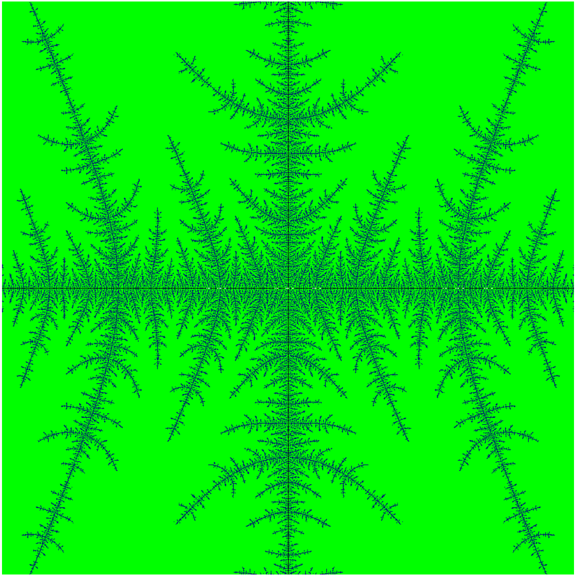

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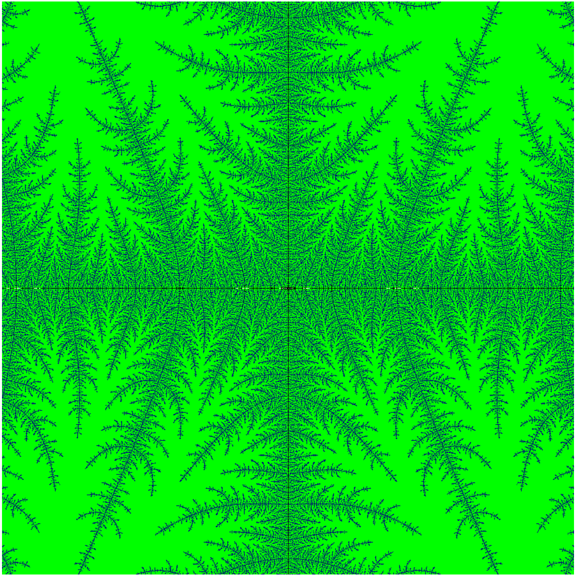

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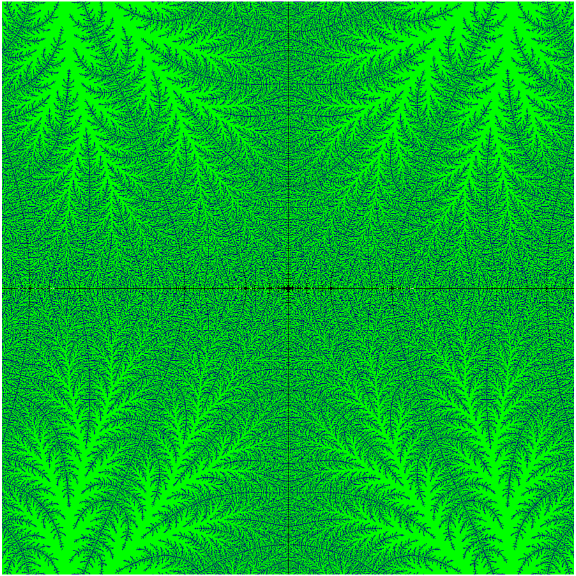

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Theorem (Shishikura, 1998): For a generic $c \in \partial\mathbb{M}$, we have $\dim_H(J_c) = 2$.

Conjecture: For any $c \in \mathbb{R}$, we have $\dim_H(J_c) < 2$.

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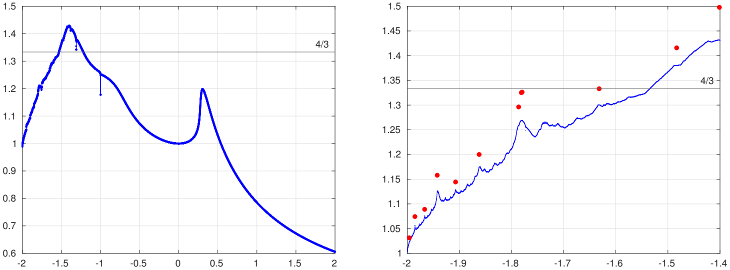

Figure: (a) A piecewise constant function bounding the hyperbolic dimension from below over the domain $[-2, 2]$; (b) The subdomain $[-2, -1.4]$ where red points have been computed with higher accuracy.

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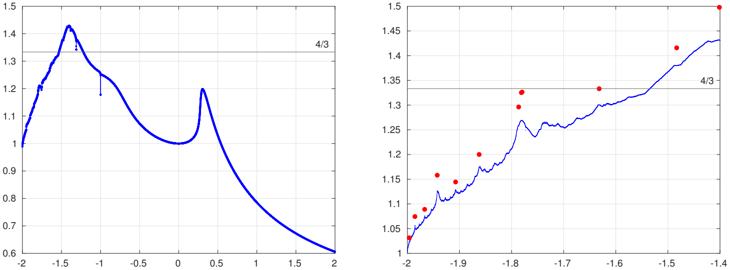

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Corollary: Figure (a) + (L. Jaksztas-M. Zinsmeister) $\Longrightarrow$ For all parameters $c \in (c_F, -3/4)$ the mapping $c \mapsto \dim_H(J_c)$ is C^1-smooth.

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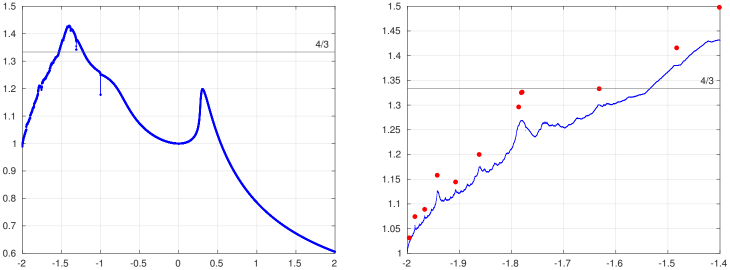

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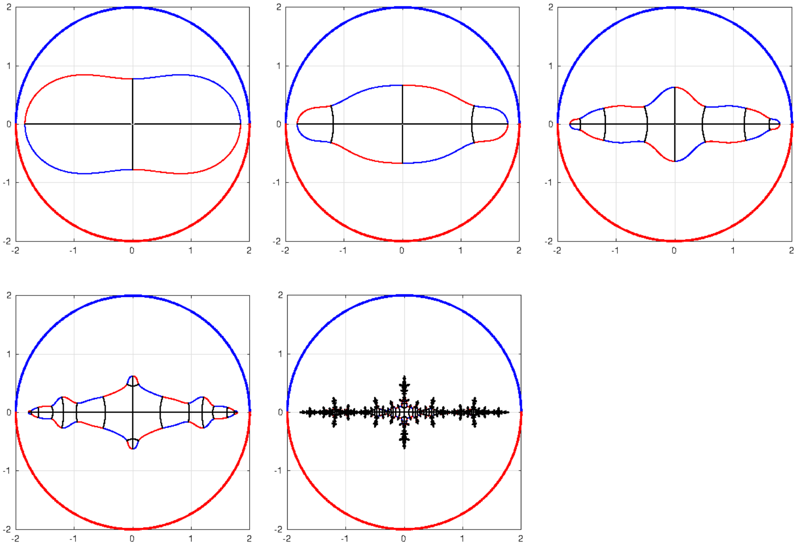
Theorem (Dobbs, Graczyk, Mihalache, 2022): There exist $K > 0$ and $c_0 > -2$ such that $\dim_H(J_c) \geq 1 + K(c + 2)^{1/2}$, for all $c \in [-2, c_0]$.

Dynamical partitions.



McMullen's matrices.

$$m_{i,j}^{(k)}(t) = \begin{cases} \max_{z \in P_i^{(k)} \cap f_c^{-1}(P_j^{(k)})} |f_c'(z)|^{-t} & \text{if } P_i^{(k)} \cap f_c^{-1}(P_j^{(k)}) \neq \emptyset, \\ 0 & \textit{otherwise}. \end{cases}$$

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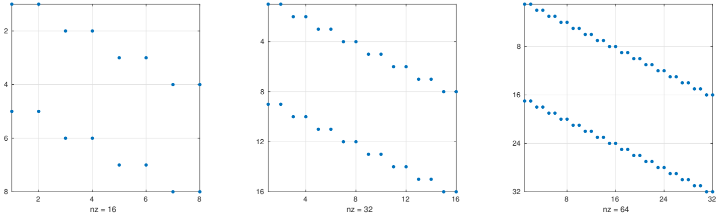

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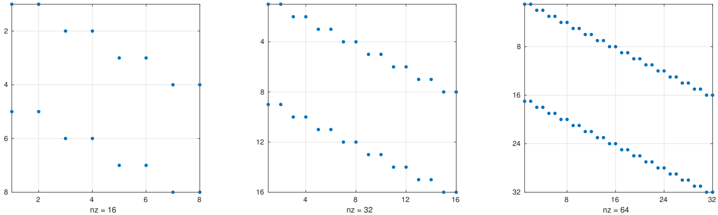

Figure: The sparse, banded structure of the McMullen matrices $M^{(k)}$.

Lemma: For any $k \in \mathbb{N}$, we have

$$\inf\{t > 0 \colon \rho(M^{(k)}(t)) = 1\} \leq \dim_H(J_c).$$

Hyperbolic dimension and geometric tree pressure.

- $\dim_{hyp}(J_c) := \sup\{\dim_H(X) \colon X \subset J_c \text{ hyperbolic for } f\}$
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$$f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \qquad \text{– rational map}$$

For every positive integer $k \in \mathbb{N}$, a point $z \in \widehat{\mathbb{C}} \setminus PC(f)$ and a real number $t > 0$, consider the pressure function

$$P(z,t,f) = \limsup_{k\to\infty} \frac{1}{k} \log \left(\sum_{w\in\widehat{\mathbb{C}}\colon\, z=f^{\circ k}(w)} |(f^{\circ k})'(w)|^{-t} \right).$$

Theorem (F. Przytycki):

(1) $P(z,t,f)$ is independent of $z \in \widehat{\mathbb{C}} \setminus PC(f)$;

(2) $t_f = \inf\{t > 0\colon P(z,t,f) \le 0\} = dim_{hyp}(J(f))$.

Circular complex arithmetic. Set-valued computations.

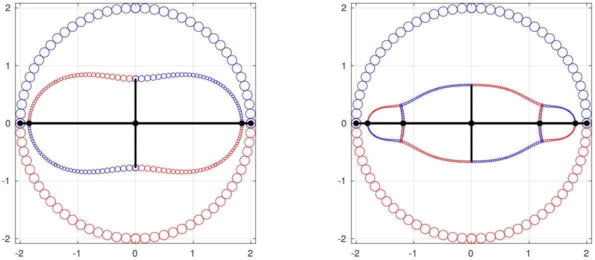

Figure: The two first levels of tiles for the parameter set $\mathbf{c} = \langle c_F, 10^{-10} \rangle$ using 39 discs to cover each initial half-disc.

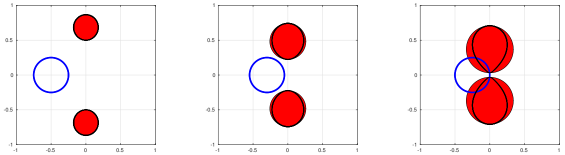

Figure: Complex square root: disk enclosures (red discs) vs the true images.

Bounding the spectral radius.

Theorem (Collatz):
$v^{(0)}$ – an arbitrary positive n-dimensional vector
$v^{(j+1)} := [v_1^{(j+1)} \ \dots \ v_n^{(j+1)}]^T = M^{j+1} v^{(0)}$

$$\underline{\lambda}^{(j)} = \min_{1 \le i \le n} \left(\frac{v_i^{(j+1)}}{v_i^{(j)}} \right) \qquad \text{and} \qquad \overline{\lambda}^{(j)} = \max_{1 \le i \le n} \left(\frac{v_i^{(j+1)}}{v_i^{(j)}} \right).$$

Then
$$\underline{\lambda}^{(0)} \le \underline{\lambda}^{(1)} \le \cdots \le \rho(M) \le \cdots \le \overline{\lambda}^{(1)} \le \overline{\lambda}^{(0)}.$$

Lemma: If A and B are irreducible matrices such that $0 \le A \le B$, then $\rho(A) \le \rho(B)$.

Computer output.

- Individual parameters (red points) were computed for $k = 28$; the period-doubling parameter – with 199 initial disks; other parameters – with 39 initial disks.
- Run time (on 12 cores) for period-doubling – ≈ 20 hours.
- Construction of the partition is the most time consuming (despite benefiting the most from parallelization).
- For estimates on intervals we used $k = 17$ or $k = 18$ with 39 initial disks.