The beginnings of validated numerics

In memory of Hans Koch and Oscar Lanford

Some ramblings about the ``history'' of validated numerics

Although ``everybody'' might know how this subject evolved, I will try to reconstruct how the early issues and difficulties came up, and how Koch and Lanford (and many others) worked to make this subject a common modern tool in analysis

Hans and Susan Koch 2014 (July 20, 1951 – Oct. 18, 2025) (Master ETHZ 1974, PhD 1979)

J.-P. Eckmann, H. Koch, P. Wittwer: A computer assisted proof of universality for area-preserving maps, Mem. Amer. Math. Soc., Vol. 47, No. 289 (1984), 1–121.

Oscar and Regina Lanford (1940–2013)

A computer-assisted proof of the Feigenbaum conjectures, Bulletin AMS, Vol 6, no3 1982

Pre-History

Paul S. Dwyer Errors of matrix computations 1953
Ramon E. Moore 1929–2015 "The automatic analysis and control of error in digital computation based on the use of interval numbers" (after PhD Stanford 1962) (Student of Forsythe)
R.E. Boche An operational interval arithmetic (1963) (speed, machine code)
Ulrich Kulisch 1933– (interval-arithmetics, hardware implementation, scalar product)
Issues: Are there validated error bounds?
Optimizing validated error bounds (best bounds)

JPE: IBM/360 (~ 1962) IBM 1620 (<1964) (20K decimal digits), tape or card reader, typewriter output ~ 850'000 $
Rutishauser, Wilkinson,. . . (Linear Algebra)

Setting up a machinery

IEEE 754 1985 William Kahan (1933–) (UC Berkeley) (Turing award 1989)

(random errors in summation Lanford)

i8086

Motorola 68000

DEC

https://people.eecs.berkeley.edu/~wkahan/ieee754status/754story.html

Arithmetics should give same results on different computers

GMP (GNU Multiple Precision Arithmetic Library)

MPFR Multiple-precision floating-point computations with correct rounding

MPFI (Inria) (Arbitrary precision)

(Tonight's discussion?)

Rounding

Probably known to everybody here:

(Round to nearest, ties away from zero (or ties to away) rounds to the nearest value; if the number falls midway, it is rounded to the nearest value above (for positive numbers) or below (for negative numbers).

Round toward 0 directed rounding towards zero (also known as truncation)).

Round toward $+\infty$ directed rounding towards positive infinity (also known as rounding up or ceiling).

Round toward $-\infty$ directed rounding towards negative infinity (also known as rounding down or floor).

Table Maker's Dilemma (Transcendental functions)

In practice

BEST bounds? $\Rightarrow$ at least rigorous bounds

I guess best today is multiplication

To-up $x > 0 \rightarrow x * 1.00000001$ (fast?)

$(x, \varepsilon) \Leftrightarrow y \in |y - x| < \varepsilon$

Compilers,

Speed? (compare to rest of research!)

Hardware

Mistakes happen:

Kahan wrote a program to check numerics

Shalom Eliahou : $3x + 1$ arithmetic error

intel pentium : mistake in a lookup table for floating point division

$$\frac{4{,}195{,}835}{3{,}145{,}727} = 1.333820449136241002$$

Rare $\sim 1/10^8$

One should check (Testfloat)

On my laptop 80-bit is wrong

For the TI-34 II and the TI-40 College II, the following functions and ranges will give incorrect results for products with date codes earlier than 1299.
The result for the range limits shown will be correct and the values between these limits will be incorrect. The log() and ln() Function
Also Y^X if Y is inside the range shown and X is any value other than an integer in the range 0,9
Also X ROOT Y if Y is inside the range shown and X is not equal to one or negative one. Also CUBE ROOT X if X is inside the range shown.

Entry Value between 1.221402765935 and 1.222222222223
Entry Value between 1.020201346154 and 1.020202020203
Entry Value between 1.002002001340 and 1.002002002003
Entry Value between 1.000200020001 and 1.000200020003
Entry Value between 0.9801980198019 and 0.9801986674200
Entry Value between 0.9980019980019 and 0.9980019986620

Office 2007, Microsoft Excel would confidently state that 77.1 × 850 = 100,000

Constructive Analysis

"Don't use Zorn's Lemma!"

Errett Bishop (1928–1983) Foundations of Constructive Analysis (1967)

L.E.J. Brouwer (1881–1966) (1927)

Douglas S. Bridges (2006) Techniques of Constructive Analysis

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Example: Intermediate value theorem

Nonconstant analytic functions have isolated zeros, and they can be found constructively.

Differentiable functions (Newton algorithm)

Continuous functions : find a sequence of c_n for which $|f(c_n)| < 1/n$

There exist irrational numbers x and y such that x^y is rational (but we do not know which) $u = \sqrt{2}^{\sqrt{2}}$. Either u is rational, or $u^{\sqrt{2}} = 2$ is rational.

Outlook?

Has everything conceptual been said?
Are there corners of analysis which have been overlooked? (relation to Bridges/Bishop?)
What new problems are waiting for "us"?