On attractors of Fibonacci maps

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Wild attractors

One of the key objects in dynamical systems is an *attractor*. Informally speaking, an attractor of is the set attracting "large" number of orbits of dynamical system. Let $f : X \to X$, where X is a topological space. For $x \in X$ let $\omega(x)$ be the ω-limit set of x. For $A \subset X$ let $B(A) = \{x \in X : \omega(x) \subset A\}$ be the basin of attraction of A. Let μ be a measure on X. A subset $A \subset X$ is called a

- *topological attractor* if $B(A)$ is a residual set and is minimal with this property,
- *metric attractor* if $\mu(B(A)) > 0$ and A is minimal with this property.

One of possible exotic situations is when metric and topological attractors of a system do not coincide. A metric attractor which is not a topological attractor is called a *wild attractor*. For a smooth map f with critical points a natural candidate for a metric attractor is the closure of the postcritical set $\mathcal{P}(f)$ of f (closure of the union of forward orbits of critical points).

Fibonacci maps

First polynomial maps on a segment with a wild attractor were Fibonacci polynomial maps (Bruin-Keller-Novicki-van Strien, 1996). Let $u_1 = 1, u_2 = 2, u_{n+1} = u_n + u_{n-1}, n \geqslant 2$, be the Fibonacci sequence.

Definition

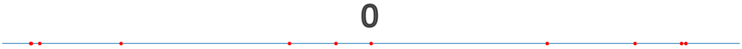
An even unimodal map $f : I \to \mathbb{R}$ with a minimum at 0 is called a Fibonacci map if the critical orbit $f^n(0)$ satisfies:

$$|f^{u_n}(0)| \geqslant |f^{u_{n+1}}(0)| \ \text{ for all } \ n \in \mathbb{N} \ \text{ and } \ |f^3(0)| < |f^4(0)|.$$

For any $d \geqslant 2, d \in \mathbb{R}$, there exists a unique $c_d \in \mathbb{R}$ such that the map $P_d(z) = |z|^d + c_d$ has Fibonacci combinatorics of the postcritical set. Notice that when d is a positive even integer $P_d(z)$ is a polynomial. A Fibonacci map has a unique critical point at the origin.

Attractors of Fibonacci maps

Example: $P_3(z) + |z|^3 + c_3, \; c_3 \approx -1.38704075.$



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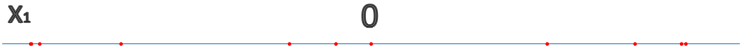

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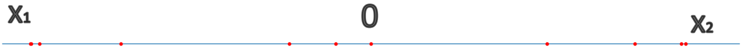

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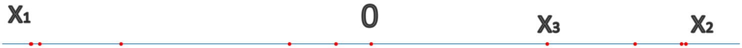

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Observe that P_d is non-renormalizable in the classical sense and does not have non-repelling periodic orbits. According to classification of topological attractors (Guckenheimer, 1979), it has a unique topological attractor, which is a cycle of segments. Thus, $\overline{\mathcal{P}(P_d)} = \omega(0)$ cannot be a topological attractor.

Theorem (Bruin-Keller-Novicki-van Strien)

There exists $d_0 > 0$ such that for any $d \geqslant d_0$ the Fibonacci map $|z|^d + c_d$ has a wild attractor.

Renormalization

Roughly speaking, renormalization is a procedure of going from a larger to a smaller scale in space while simultaneously going from a smaller to a larger scale in time. It appeared as a collection of useful techniques in physics (e.g. quantum field theory and electrodynamics). Later, renormalization was introduced in various subareas of dynamical systems.

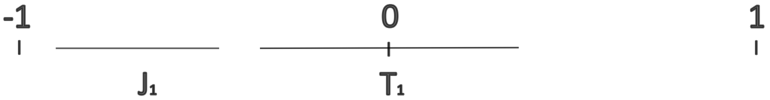
In 1-dimensional dynamics: renormalization of real 1-dimensional maps, renormalization of quadratic-like maps, parabolic renormalization (for maps with periodic points with multiplier $e^{2\pi i\theta}, \theta \in \mathbb{Q}$), Siegel, cylinder, pacman renormalizations (with $\theta \in \mathbb{Q}$).

Fibonacci renormalization

The Fibonacci polynomials P_d are not renormalizable in the usual sense, but can be studied using an appropriate generalized renormalization (Lyubich-Milnor, 1993). Let $F : J_1 \cup T_1 \to [-1, 1]$ be a C^2 map, wher $J_1 = [a, b]$, $T_1 = [\alpha, \beta]$, and $-1 < a < b < \alpha < \beta < 1$. Assume that

(*i*) $F|_{J_1}$ is a diffeo onto $[-1, 1]$;

(*ii*) $F|_{T_1}$ is unimodal with minimum at $x_0 = 0$ and $F(\partial T_1) = 1$.



The class $\mathcal{A}$ of such maps is union of two disjoint components $\mathcal{A}^+$ and $\mathcal{A}^-$ (F_{J_1} is orientation preserving or reversing, respectively). Assume further that $F(x_0) \in J_1$ and $F^2(x_0) \in T_1$. Then there exists an interval $T_2 \ni x_0$, mapped unimodally into T_1 by F^2, such that $F^2(\partial T_2)$ is an endpoint of T_1. Also, there is $J_2 \subset T_1$, $J_2 \cap T_2 = \emptyset$, such that $F|_{J_2}$ is a diffeo onto T_1.

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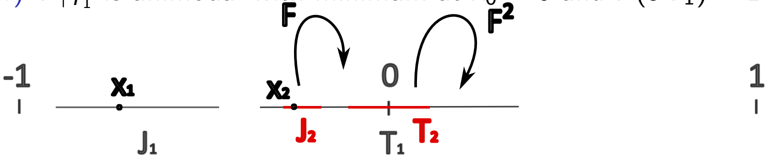

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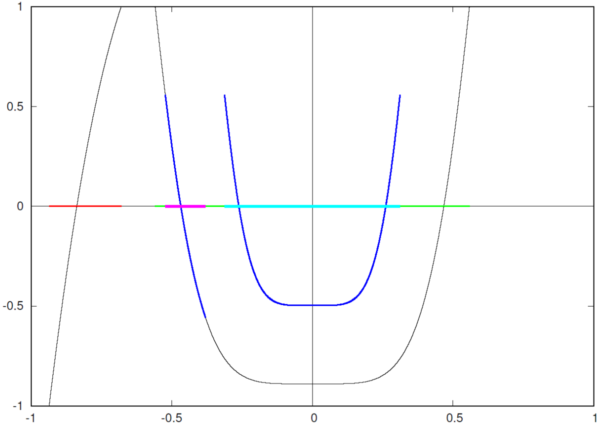


FIGURE 1.1. A renormalizable Fibonacci map of degree $d = 5.0$. The original map is defined over the intervals J_1 and T_1. Its prerenormalization, in blue, is defined over the intervals $J_2 \subset T_1$ and $T_2 \subset T_1$.

Fibonacci renormalization

The map from $J_2 \cup T_2$ onto T_1 given by $(F|_{J_2}, F^2|_{T_2})$ is called the prerenormalization of F, denoted $\mathcal{P}F$ or F_1. $\mathcal{P}F$ rescaled by the affine transformation s_{T_1} that maps T_1 onto $[-1,1]$, is called the renormalization of F, denoted $\mathcal{R}F$. The operator $\mathcal{R}$ interchanges the components $\mathcal{A}^{\pm}$.
Lyubich and Milnor showed that $F \in \mathcal{A}$ is infinitely renormalizable iff its critical orbit has Fibonacci combinatorics. Furthermore, the restriction of a Fibonacci polynomial P_2 to $[P_2^5(0), P_2^2(0)] \cup [P_2(0), P_2^4(0)]$ can be extended to a smooth map of class $\mathcal{A}^-$. Similar is true for P_d, $d \geqslant 2$. The operator $\mathcal{R}$ does not have fixed points, since it switches the orientation of the diffeomorphic part.

Conjecture

For every $d > 2$ the operator $\mathcal{R}$ has a unique period 2 periodic point having critical point of degree d.

Periodic points of renormalization

We construct an algorithm given $d > 2$ to show existence of a period 2 periodic point F of the Fibonacci renormalization $\mathcal{R}$ using computer assistance. In particular, we show.

Proposition

Let $d = 3.8$ or $d = 5.1$. Then there exists a map $F_d \in \mathcal{A}$ having critical point of degree d such that $\mathcal{R}^2 F_d = F_d$. Moreover, the operator $\mathcal{R}$ is hyperbolic near F_d.

Using computer estimates, we obtain accurate approximations of F_d for the above values of d.

Theorem

The map $F_{3.8}$ does not have a wild attractor, but $F_{5.1}$ has a wild attractor.

One has $P_{3.8}(z) \approx |z|^{3.8} - 1.26835425$,
$P_{5.1}(z) \approx |z|^{5.1} - 1.17912392$.

Notations

Fix $d > 2$. Let $\{F, G\} = \{(f, g), (f, -g)\}$ be the period two periodic cycle with a critical point of degree d. Here $f : T_1 = [-\lambda, \lambda] \to T_0 = [-1, 1]$ is an even unimodal map with $f(\lambda) = 1$, and $g : J_1 = [a, b] \to T_0$ is an increasing homeomorphism, $a < f(0) < b$.

For $n \geqslant 0$, let T_{n+1} and J_{n+1} be the domains of the n-th prerenormalization $F_n = (f_n, g_n)$ of F:

$$T_{n+1} = \lambda^n T_1,\ J_{n+1} = \pm\lambda^n J_1,$$
$$f_n(z) = \pm\lambda^n f(z/\lambda^n),\ g_n(z) = \pm\lambda^n g(\pm z/\lambda^n),$$

where λ is the scaling ratio (depending on d). Introduce the sets

$$X_n = \{x \in T_1 : F^k(x) \in T_n \text{ for some } k \geqslant 0\},$$
$$Y_n = \{y \in T_n : F^k(x) \notin T_n \text{ for all } k \geqslant 1\}$$

and the quantities $\eta_n = |X_n|/|T_1|$, $\xi_n = |Y_n|/|T_n|$, where $|\cdot|$ is the Lebesgue measure of a set.

Constructive trichotomy for Fibonacci maps

Let $\mathcal{P} = \{F^k(0) : k \in \mathbb{Z}_+\}$ be the postcritical set of F, and $\mathcal{C} = \overline{\mathcal{P}}$ be the metric attractor of F. Recall that F has a wild attractor iff the basin of attraction $B(\mathcal{C})$ has positive Lebesgue measure. We show:

Theorem

There exists a constructive constant $C > 1$ (depending on d), such that one of the following holds:

(1) *There exists $n \in \mathbb{N}$ such that $\eta_n/\xi_n < C^{-1}$. Then $\xi_n \to 0$ exponentially fast, $\inf \xi_n > 0$, and F does not have a wild attractor ($B(\mathcal{C})$ has zero Lebesgue measure).*

(2) *For all n one has $C^{-1} \leqslant \eta_n/\xi_n \leqslant C$. Then $\eta_n \to 0$, $\xi_n \to 0$, and F does not have a wild attractor ($B(\mathcal{C})$ has zero Lebesgue measure).*

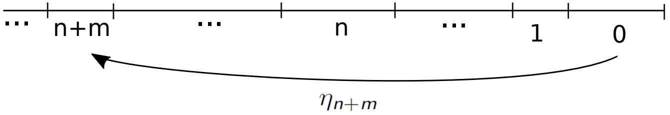
(3) *There exists $n \in \mathbb{N}$ such that $\eta_n/\xi_n > C$. Then $\inf \eta_n > 0$, $\xi_n \to 0$, and F has a wild attractor ($B(\mathcal{C})$ has positive Lebesgue measure).*

Recursive estimates

Proposition

There exists constructive constant C such that for all n, m one has

$$\frac{\eta_n\eta_m}{\eta_m + C\xi_n} \leqslant \eta_{n+m} \leqslant \frac{\eta_n\eta_m}{\eta_m + C^{-1}\xi_n}.$$



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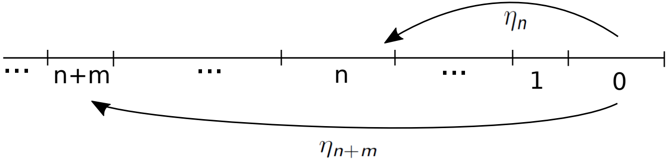

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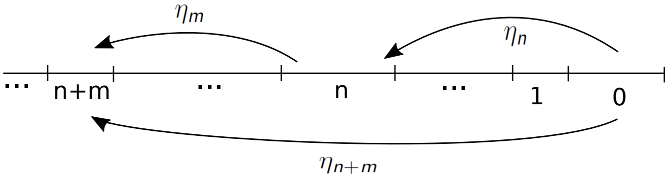

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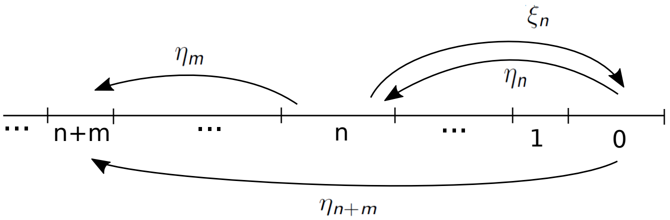

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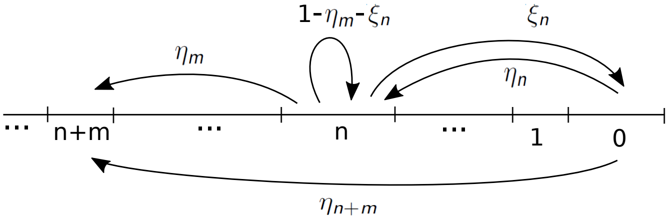

$$\eta_{n+m} \lesssim \eta_n\eta_m \sum_{j\geqslant 0}(1-\eta_m-\xi_n)^j = \frac{\eta_n\eta_m}{\eta_m+\xi_n}.$$

Absence of a wild attractor

Proposition

Assume that $\eta_n/\xi_n < C^{-1}$ for some n. Then η_n converges to zero exponentially fast when $n \to \infty$ and Fibonacci map under consideration does not have a wild attractor.

Proof.

Fix n and γ such that $C\eta_n/\xi_n < \gamma < 1$. Then we have for all m:

$$\eta_{n+m} \leqslant \frac{\eta_{m+1}}{\frac{\eta_{m+1}}{\eta_n} + C^{-1}\frac{\xi_n}{\eta_n}} \leqslant \gamma\eta_{m+1}.$$

Therefore, $\eta_k (= |X_k|/|T_1|)$ converges to zero (exponentially fast) when $k \to \infty$. Recall that $\eta_k = |X_k|/|T_1|$, and so $\cap_{k\in\mathbb{N}} X_k$ has zero Lebesgue measure. Notice that $B(\mathcal{C}) \subset \cap_{k\in\mathbb{N}} X_k$. Thus, the Fibonacci map F does not have a wild attractor. $\square$

Distortion bounds

Proposition

There exists $\epsilon = \epsilon_d > 0$ such that the following is true. Let F be as before (a period 2 periodic points under $\mathcal{R}$) and $n \in \mathbb{N}$. Given $x \in T_1 \cup J_1$ let $k \in \mathbb{N}$ be the minimal number such that $F^k(x) \in T_n$. Then the inverse branch ϕ of F^k such that $\phi(F^k(x)) = x$ continues to an analytic map on the segment $(1+\epsilon)T_n$.

Corollary

There exists a constant C depending only on d such that in the settings as above one has:

$$|\phi'(x)|/|\phi'(y)| \leqslant C \text{ for all } x, y \in T_n.$$

Real renormalization

Denote $\mathcal{D}^\omega$ the Banach space of orientation-preserving C^ω-diffeomorphisms of $[-1,1]$ onto itself. We consider a map $F=(g,f)$ on $J\cup T$ given by

$$F(x)=\begin{cases}\pm\psi(s_J^{-1}(x)), & x\in J,\\ \phi(p_v(s_T^{-1}(x))), x\in T,\end{cases} \tag{1}$$

where $\pm$ depends on the orientation of g,
$J=[i,j]$, $T=[-t,t]\subset[-1,1]$ are disjoint, s_I denotes an orientation-preserving affine map of I onto $[-1,1]$,
$(\psi,\phi)\in\mathcal{D}^\omega\times\mathcal{D}^\omega$, and

$$p_v(x)=v+(1-v)|x|^d,\quad v\in[-1,1]. \tag{2}$$

Restrict t,i,j to appropriate closed intervals A,B,C correspondingly. Then the set of such maps F is isomorphic to the product Banach space

$$\mathcal{B}=[-1,1]\times A\times B\times C\times\mathcal{D}^\omega\times\mathcal{D}^\omega. \tag{3}$$

Interval arithmetic in $\mathbb{R}$

We will endow the Banach space $\mathcal{B}$ with the following l_1-norm:

$$\|F\|_1 = |v| + |i| + |j| + \|\psi\|_1 + \|\phi\|_1. \tag{4}$$

In our computations, we use interval arithmetic in $\mathbb{R}$ and $\mathcal{D}^\omega$. Denote by $\mathfrak{R}$ the collection of real representable numbers (in the standard IEEE floating format). Define the *standard sets* in $\mathfrak{R}$ as:

$$\mathrm{S}(\mathfrak{R}) = \{I[x, y] : x, y \in \mathfrak{R}\}. \tag{5}$$

Roughly speaking, result of a standard operation on $I_1, I_2 \in \mathrm{std}$ gives a segment $I \in \mathrm{S}$ containing all possible values of this operation on $(x_1, x_2) \in I_1 \times I_2$. For instance,

addition: $I[x_1, y_1] + I[x_2, y_2] = I[x_1 + x_2, y_1 + y_2]$;

multiplication: $I[x_1, y_1] \cdot I[x_2, y_2] = I[x_3, y_3]$, where x_3, y_3 are the minimum and the maximum of $x_1 x_2, x_1 y_2, y_1 x_2, y_1 y_2$.

Approximating periodic points of renormalization

Let $\mathcal{R}$ be an operator analytic and hyperbolic on some neighborhood $\mathcal{N}$ in a Banach space and $Z_0 \in \mathcal{N}$ be an approximation of its hyperbolic fixed point. Set

$$M \equiv [\mathbf{I} - D_a]^{-1},$$

where D_a is some finite-dimensional approximation of $D\mathcal{R}[Z_0]$. Define, for all z, such that $Z_0 + Mz \in \mathcal{N}$,

$$N[z] = z + \mathcal{R}[Z_0 + Mz] - (Z_0 + Mz). \qquad (6)$$

Notice, that if z^* is a fixed point of N, then $Z_0 + Mz^*$ is a fixed point of $\mathcal{R}$. The linear operator $\mathbf{I} - D_a$ is indeed invertible since $D\mathcal{R}$ is hyperbolic at Z_0.

Approximating periodic points of renormalization

If Z_0 is a reasonably good approximation of the true fixed point of $\mathcal{R}$, then the operator N is expected to be a strong contraction in a neighborhood of 0:

$$\begin{aligned} DN[z] &= \mathbf{I} + D\mathcal{R}[Z_0 + Mz] \cdot M - M \\ &= \left[M^{-1} + D\mathcal{R}[Z_0 + Mz] - \mathbf{I}\right] \cdot M \\ &= [\mathbf{I} - D_a + D\mathcal{R}[Z_0 + Mz] - \mathbf{I}] \cdot M \\ &= [D\mathcal{R}[Z_0 + Mz] - D_a] \cdot M. \end{aligned}$$

The last expression is typically small in a small neighborhood of 0, if the norm of M is not too large.

Case $d = 3.8$: the existence Theorem

There exist two polynomials of degree 300, $\psi_{3.8}$ and $\phi_{3.8}$, such that the Newton operator for $\mathcal{R}$ defined above with

$$\begin{aligned} Z_0 = (& -0.83700583021901265, -0.89842138158302138, \\ & -0.64345222855602410, 0.44908966263509253, \psi_{3.8}, \phi_{3.8}), \end{aligned}$$

is a contraction on the ball $B_\delta(0) = \{Q \in \mathcal{B} : \|Q\|_1 < \delta\}$, where $\delta = 4.28577601155835506 \times 10^{-12}$, with

$$\begin{aligned} \epsilon = \|N(0)\|_1 \leq 3.89968810858366172 \times 10^{-13} \\ D = \|DN|_{B_\delta}\|_1 \leq 8.35798206185280828 \times 10^{-7}. \end{aligned}$$

In particular, $B_\delta(0)$ contains a fixed point of the operator N.

Thank you!