Rigorous enclosure of Lyapunov exponents of stochastic flows

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Stochastic Differential Equations (SDEs)

Consider a stochastic differential on $\mathcal{M} = \mathbb{T}^{d_1} \times \mathbb{R}^{d_2}$, with $d = d_1 + d_2$

$$\mathrm{d}\varphi_t(\omega, x) = V(\varphi_t(\omega, x))\mathrm{d}t + \sigma \mathrm{d}B_t(\omega), \qquad \varphi_0(\omega, x) = x$$

with smooth vector field V on $\mathcal{M}$, $\sigma > 0$ and $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion on a probability space $(\Omega, \mathbb{P})$. Formally,

$$\varphi_t(\omega, x) = x + \int_0^t V(\varphi_\tau(\omega, x))\mathrm{d}\tau + \sigma B_t(\omega) \qquad \text{for all } t \geq 0$$

You can think of Euler's integration method

$$\varphi_{t+\Delta t}(\omega) \approx \varphi_t(\omega) + V(\varphi_t(\omega))\Delta t + \sigma \Delta B_t(\omega),$$

with $\Delta B_t \sim \mathcal{N}(0, \Delta t\, \mathrm{Id})$.
Note that $(\varphi_t)_{t\geq 0}$ is a Markov process.

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For $\mathbb{P}$-almost all $\omega \in \Omega$ and $t \geq 0$,

$$\varphi_t(\omega, \cdot) : \mathcal{M} \to \mathcal{M} \quad \text{is a } \textbf{diffeomorphism}.$$

We are interested in this diffeomorphism as $t \to \infty$, e.g. given $x \neq y$,

$$|\varphi_t(\omega, x) - \varphi_t(\omega, y)| \to ?$$

All pathwise statements in this talk are $\mathbb{P}$-almost surely!

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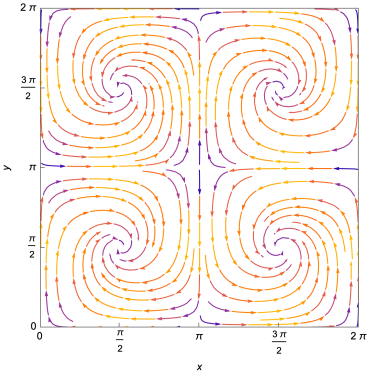
Example: the cellular flow with sinks

Consider the cellular flow with sinks

$$\begin{cases} \mathrm{d}x_t &= (\kappa \cos x_t - \cos y_t) \sin x_t \mathrm{d}t + \sigma \mathrm{d}B_t^1 \\ \mathrm{d}y_t &= (\kappa \cos y_t + \cos x_t) \sin y_t \mathrm{d}t + \sigma \mathrm{d}B_t^2 \end{cases}$$

with $\kappa, \sigma > 0$.

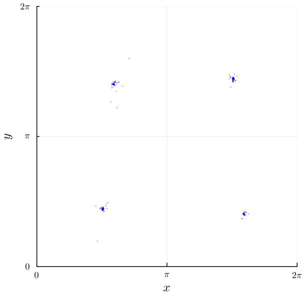


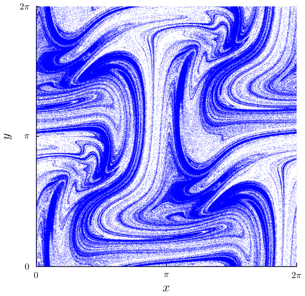
Synchronisation vs Chaos

$$\begin{cases} \mathrm{d}x_t &= (\kappa \cos x_t - \cos y_t) \sin x_t \mathrm{d}t + \sigma \mathrm{d}B_t^1 \\ \mathrm{d}y_t &= (\kappa \cos y_t + \cos x_t) \sin y_t \mathrm{d}t + \sigma \mathrm{d}B_t^2 \end{cases}$$

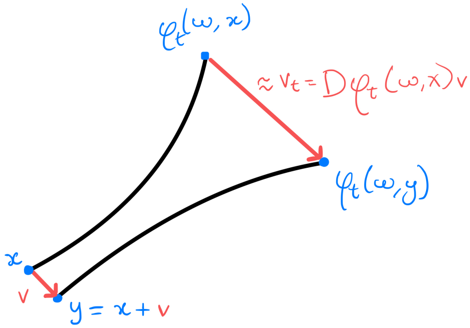
$\kappa = 1/2, \sigma = 1/4$: Synchronisation

$\kappa = 1/2, \sigma = \sqrt{2}$: Chaos





The Lyapunov exponent



Define the Lyapunov exponent

$$\lambda(\omega, x, v) = \lim_{t \to \infty} \frac{1}{t} \log \|D\varphi_t(\omega, x)v\|$$

$\lambda(\omega, x, v)$ is well-defined and **constant** under some ergodicity assumptions.

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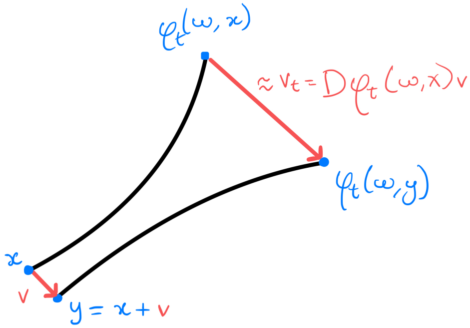

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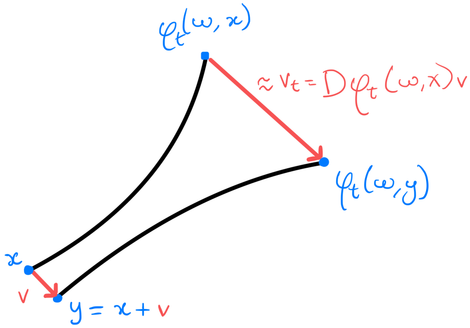

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Lyapunov exponent for our example

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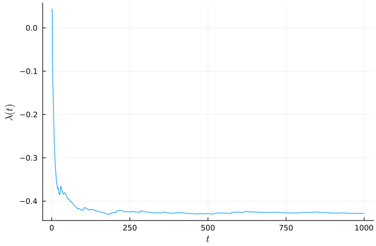
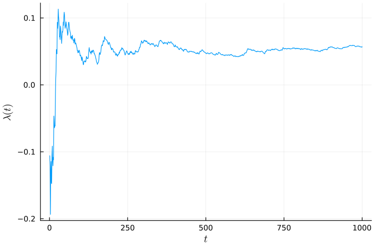
$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log \| D\varphi_t(\omega, x) v \|$$

$\kappa = 1/2, \sigma = 1/4 : \lambda \approx -0.43 < 0$ $\qquad$ $\kappa = 1/2, \sigma = \sqrt{2} : \lambda \approx 0.056 > 0$





Noise-induced chaos

Recall the cellular flow with sinks on $\mathbb{T}^2$

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The vast majority of results on the positivity of the Lyapunov exponent for stochastic flows are:

- for Hamiltonian/volume-preserving systems: not true if $\kappa \neq 0$.
- in the small noise limit $\sigma \approx 0$: clearly, $\lambda < 0$ for small noise and we want to show $\lambda > 0$ for $\sigma = \sqrt{2}$.

Idea: use **computer-assisted methods!** They work far from perturbative regimes.

Some other works on sign of Lyapunov exponents: Baxendale, Goukasian (AOP, 2002); Pollicott (Invent. Math. 2010); Bedrossian, Blumenthal, Punshon-Smith (Invent. Math. 2022, ICM 2022); Baxendale (PTRF, 2024)

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How to compute Lyapunov exponents?

Recall

$$\varphi_t(\omega, x) = x + \int_0^t V(\varphi_\tau(\omega, x))\mathrm{d}\tau + \sigma B_t(\omega) \qquad \text{for all } t \geq 0$$

For $v \in \mathbb{S}^{d-1}$, write

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Since

$$D\varphi_t(\omega, x) = \mathrm{Id} + \int_0^t DV(\varphi_\tau(\omega, x))D\varphi_\tau(\omega, x)\mathrm{d}\tau$$

the chain rule shows that

$$\frac{\mathrm{d}s_t}{\mathrm{d}t} = DV(\varphi_t)s_t - Q(\varphi_t, s_t)s_t,$$

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The projective process

We have

$$\begin{cases} \mathrm{d}\varphi_t & = V(\varphi_t)\mathrm{d}t + \sigma B_t(\omega) \\ \mathrm{d}s_t & = [DV(\varphi_t)s_t - Q(\varphi_t, s_t)s_t]\,\mathrm{d}t \end{cases}$$

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Then

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\log\varrho_t = \lim_{t\to\infty}\frac{1}{t}\int_0^t Q(\varphi_\tau, s_\tau)\mathrm{d}\tau.$$

Thus, λ is a **time-average** over the Markov process $(\varphi_t, s_t)_{t\geq 0}$ on $\mathcal{M}\times\mathbb{S}^{d-1}$ which is called the **projective process**.

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The Furstenberg–Khasminskii formula

$$\begin{cases} \mathrm{d}\varphi_t &= V(\varphi_t)\mathrm{d}t + \sigma \mathrm{d}B_t(\omega) \\ \mathrm{d}s_t &= [DV(\varphi_t)s_t - Q(\varphi_t, s_t)s_t]\,\mathrm{d}t \end{cases}$$

where $Q(x, s) = \langle DV(x)s, s\rangle$ and

$$\lambda = \lim_{t\to\infty} \frac{1}{t}\int_0^t Q(\varphi_\tau, s_\tau)\mathrm{d}\tau.$$

The projective process $(\varphi_t, s_t)_{t\geq 0}$ is said to be **ergodic** if there exists a unique stationary probability density $p : \mathcal{M} \times \mathbb{S}^{d-1} \to \mathbb{R}_+$ such that for all measurable $A \subset \mathcal{M} \times \mathbb{S}^{d-1}$

$$\int_{\mathcal{M}\times\mathbb{S}^{d-1}} \mathbb{P}\left((\varphi_t(\cdot, x), s_t(\cdot, x, s)) \in A\right) p(x, s)\mathrm{d}x\mathrm{d}s = \int_A p(x, s)\mathrm{d}x\mathrm{d}s.$$

Under this assumption, by **Birkhoff's ergodic theorem**,

$$\lambda = \lim_{t\to\infty} \frac{1}{t}\int_0^t Q(\varphi_\tau, s_\tau)\mathrm{d}\tau = \int_{\mathcal{M}\times\mathbb{S}^{d-1}} Q(x, s)p(x, s)\mathrm{d}x\mathrm{d}s$$

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$$\begin{cases} \mathrm{d}\varphi_t & = V(\varphi_t)\mathrm{d}t + \sigma \mathrm{d}B_t(\omega) \\ \mathrm{d}s_t & = [DV(\varphi_t)s_t - Q(\varphi_t, s_t)s_t]\,\mathrm{d}t \end{cases}$$

where $Q(x,s) = \langle DV(x)s, s\rangle$ and

$$\lambda = \lim_{t\to\infty} \frac{1}{t}\int_0^t Q(\varphi_\tau, s_\tau)\mathrm{d}\tau.$$

The projective process $(\varphi_t, s_t)_{t\geq 0}$ is said to be **ergodic** if there exists a unique stationary probability density $p : \mathcal{M} \times \mathbb{S}^{d-1} \to \mathbb{R}_+$ such that for all measurable $A \subset \mathcal{M} \times \mathbb{S}^{d-1}$

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Under this assumption, by **Birkhoff's ergodic theorem**,

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where $h(x, s) = DV(x)s + \langle DV(x)s, s\rangle$. The process $(\varphi_t, s_t)_{t\geq 0}$ is ergodic if there exists a unique probability solution p to the stationary **Fokker–Planck equation**

$$\mathcal{L}^* p = 0,$$

where

$$\mathcal{L}^* p := -\nabla_x \cdot (Vp) - \nabla_s \cdot (hp) + \frac{\sigma^2}{2}\Delta_x p.$$

In this talk, we assume that the existence, uniqueness and smoothness of p which are however non-trivial.

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Can we solve the Fokker–Planck equation?

Recall that we want to compute

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- Standard computer-assisted proofs techniques cannot at this stage handle a fully non-elliptic operators of this type. We would need to control quantitatively the spectral gap of $\mathcal{L}^*$.
- The problem is even harder if we are on an unbounded domain such as $\mathcal{M} = \mathbb{T}\times\mathbb{R}$ or $\mathcal{M} = \mathbb{R}^2$.

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The adjoint operator

The Fokker–Planck operator

$$\mathcal{L}^* \cdot := -\nabla_x \cdot (V \cdot) - \nabla_s \cdot (h \cdot) + \frac{\sigma^2}{2} \Delta_x \cdot$$

is the L^2-adjoint of the **generator** $\mathcal{L}$ of the projective process $(\varphi_t, s_t)_{t \geq 0}$ which can be written as

$$\mathcal{L} = V \cdot \nabla_x + h \cdot \nabla_s + \frac{\sigma^2}{2} \Delta_x$$

i.e. for all u, v regular enough

$$\int_{\mathcal{M} \times \mathbb{S}^{d-1}} u(\mathcal{L}^* v) \mathrm{d}x \mathrm{d}s = \int_{\mathcal{M} \times \mathbb{S}^{d-1}} (\mathcal{L} u) v \mathrm{d}x \mathrm{d}s$$

Its use on the computer is the main idea of this work!

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The adjoint method

$$\lambda = \int Q(\xi)p(\xi)\,\mathrm{d}\xi, \quad \xi = (x, s), \qquad \mathcal{L}^* p = 0$$

- We want to find an enclosure of λ, but without having access to p.
- Recall that $\mathcal{L}^*$ is the L^2-adjoint of the infinitesimal generator

$$\mathcal{L} = V \cdot \nabla_x + ((DV - \langle s, DVs\rangle)s) \cdot \nabla_s + \frac{\sigma^2}{2}\Delta_x$$

- For any u, $\int (\mathcal{L}u)p\,\mathrm{d}\xi = \int u(\mathcal{L}^* p)\,\mathrm{d}\xi = 0$. Therefore

$$\lambda = \int (Q - \mathcal{L}u)\,p\,\mathrm{d}\xi.$$

- This is the main idea of the **adjoint method**: pick a u such that $(Q - \mathcal{L}u)$ is better behaved or easier to control than Q itself.

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$$\lambda = \int (Q - \mathcal{L}u)\, p \,\mathrm{d}\xi \qquad \text{for any } u.$$

- Pick $\bar{\lambda} \approx \lambda$ and $\bar{u}$ such that $Q - \mathcal{L}\bar{u} \approx \bar{\lambda}$, i.e. a numerical solution to the **Poisson problem** $\mathcal{L}u = Q - \lambda$.
- We then get

$$\lambda = \int (Q - \mathcal{L}\bar{u})\, p \,\mathrm{d}\xi = \int \left(Q - \mathcal{L}\bar{u} - \bar{\lambda}\right) p \,\mathrm{d}\xi + \bar{\lambda},$$

therefore, letting $\bar{Q} := \mathcal{L}\bar{u} + \bar{\lambda}$

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Back to the cellular flow with sinks

Recall the cellular flow with sinks with $\kappa = 1/2$

$$\begin{cases} \mathrm{d}x_t &= (\kappa \cos x_t - \cos y_t) \sin x_t \mathrm{d}t + \sigma \mathrm{d}B_t^1 \\ \mathrm{d}y_t &= (\kappa \cos y_t + \cos x_t) \sin y_t \mathrm{d}t + \sigma \mathrm{d}B_t^2 \end{cases}$$

Then choosing $s_t = \frac{D\varphi_t v}{\|D\varphi_t v\|} = (\cos(\theta_t/2), \sin(\theta_t/2))$, we find that

$$\begin{aligned} \frac{\mathrm{d}\theta_t}{\mathrm{d}t} &= \frac{1}{2} \left(\sin\theta(4\cos x_t \cos y_t - \cos 2x_t + \cos 2y_t) - 4\sin x_t \sin y_t \right) \\ &=: h(x_y, y_t, \theta_t) \end{aligned}$$

and $Q(x, y, \theta) = \frac{1}{4}(\cos 2x + \cos 2y - \cos\theta(4\cos x \cos y + \cos 2y - \cos 2x))$.
We need to numerically solve $\mathcal{L}u = Q - \lambda$ where

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Numerical solution

Since our problem is on $\mathbb{T}^3$, we naturally seek a numerical solution

$$\bar{u} \in \mathcal{H}_{K,M,N} = \left\{ \sum_{k=-K}^{K} \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_{k,m,n} e^{ikx} e^{imy} e^{in\theta} \right\}$$

to the Poisson problem

$$\mathcal{L}u = Q - \lambda$$

Denoting P_0 the projection on constants in $L^2(\mathbb{T}^3)$, we solve

$$\bar{u} \approx \underset{u \in \mathcal{H}_{K,M,N}}{\operatorname{argmin}} \|P_0^{\perp}(\mathcal{L}u - Q)\|_{L^2(\mathbb{T}^3)}^2,$$

Note that so far we have done no rigorous computation.
Noting that $\mathcal{L}$ maps $\mathcal{H}_{K,M,N}$ to $\mathcal{H}_{K+2,M+2,N+2}$, we then define

$$\bar{\lambda} \overset{\text{def}}{=} P_0(Q - \mathcal{L}\bar{u}) \qquad \text{and} \qquad \bar{Q} \overset{\text{def}}{=} \mathcal{L}\bar{u} + \bar{\lambda} \approx Q,$$

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$$\bar{\lambda} \stackrel{\text{def}}{=} P_0(Q - \mathcal{L}\bar{u}) \qquad \text{and} \qquad \bar{Q} \stackrel{\text{def}}{=} \mathcal{L}\bar{u} + \bar{\lambda} \approx Q,$$

Numerical solution

Since our problem is on $\mathbb{T}^3$, we naturally seek a numerical solution

$$\bar{u} \in \mathcal{H}_{K,M,N} = \left\{ \sum_{k=-K}^{K} \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_{k,m,n} e^{ikx} e^{imy} e^{in\theta} \right\}$$

to the Poisson problem

$$\mathcal{L}u = Q - \lambda$$

Denoting P_0 the projection on constants in $L^2(\mathbb{T}^3)$, we solve

$$\bar{u} \approx \operatorname*{argmin}_{u \in \mathcal{H}_{K,M,N}} \|P_0^{\perp}(\mathcal{L}u - Q)\|^2_{L^2(\mathbb{T}^3)},$$

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Computation of the bound via the adjoint method

Recall that sine $\mathcal{L}\bar{u}$ is p-mean zero

$$\lambda - \bar{\lambda} = \int_{\mathbb{T}^3} (Q - \bar{Q}) p \,\mathrm{d}x\mathrm{d}y\mathrm{d}\theta \quad \text{and}$$

Denoting

$$Q - \bar{Q} = \sum_{k=-(K+2)}^{K+2} \sum_{m=-(M+2)}^{M+2} \sum_{n=-(N+2)}^{N+2} \epsilon_{k,m,n} e^{ikx} e^{imy} e^{in\theta}$$

where the $\epsilon_{k,m,n}$'s should be small, we have

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Result for the cellular flow with sinks

Theorem (Breden, C., Lamb, Rasmussen 2024)

Consider the cellular flow with sinks $(\varphi_t)_{t\geq 0}$ on $\mathcal{M} = \mathbb{T}^2$

$$\begin{cases} \mathrm{d}x_t &= (\kappa \cos x_t - \cos y_t) \sin x_t \mathrm{d}t + \sigma \mathrm{d}B_t^1 \\ \mathrm{d}y_t &= (\kappa \cos y_t + \cos x_t) \sin y_t \mathrm{d}t + \sigma \mathrm{d}B_t^2. \end{cases} \tag{1}$$

Then, for $\kappa = 1/2$ and $\sigma = \sqrt{2}$, we have the following bounds for λ

$$\lambda = 0.0558453099857 \pm 10^{-13} > 0.$$

Corollary

For $\kappa = 1/2$ and $\sigma = \sqrt{2}$, the system (1) displays a random strange attractor of (Lyapunov) dimension

$$\mathcal{D} = 1.475029031376 \pm 1.1 \times 10^{-12}.$$

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Some advantages of our method

- The only rigorous computation (i.e. done in interval arithmetic) is the application of $\mathcal{L}$ to $\bar{u}$, this is much cheaper than usual computer-assisted proofs which usually require a large matrix-matrix multiplication; in terms CPU time and memory.
- We can numerically solve the Poisson problem

$$\mathcal{L}u = Q - \lambda$$

  with iterative methods if $\mathcal{L}$ is sparse w.r.t. our basis
- Therefore, we can take many basis functions!
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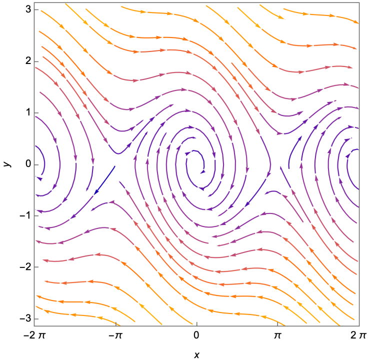
Example: the noisy pendulum

Consider the noisy pendulum (or Josephson junction)

$$\text{“}\ \ddot{x}_t = -\kappa \sin x_t - \gamma \dot{x}_t + \sigma \dot{B}_t\ \text{”}$$

with $\kappa = 2/3$, $\gamma = 1/4$. It translates to the proper SDE on $\mathbb{T} \times \mathbb{R}$

$$\begin{cases} \mathrm{d}x_t &= y_t \mathrm{d}t \\ \mathrm{d}y_t &= -(\kappa \sin x_t + \gamma y_t)\mathrm{d}t + \sigma \mathrm{d}B_t \end{cases}$$



How to extend the method to an unbounded domain?

For this example, similarly to the previous one, we find that

$$\mathcal{L} = y\partial_x - (\kappa \sin x + \gamma y)\partial_y + h(x, y, \theta)\partial_\theta + \frac{\sigma^2}{2}\partial_{yy}$$

where $h(x, y, \theta) = (1 + \gamma \sin\theta - \cos\theta + \kappa(\cos\theta + 1)\cos x)$ and

$$Q(x, y, \theta) = \frac{1}{2}(\gamma \cos\theta + \sin\theta - \gamma - \kappa \sin\theta \cos x).$$

$\mathcal{L}$ is polynomial in y, so we expect $\bar{u}$ and thus $\bar{Q}$ to be polynomial in y. Idea: use an exponentially growing weight $W \geq 0$ function such that

$$|\lambda - \bar{\lambda}| = \left| \int_{\mathbb{T}\times\mathbb{R}\times\mathbb{T}} (Q - \bar{Q}) p \,\mathrm{d}x\mathrm{d}y\mathrm{d}\theta \right| \leq \left\| \frac{Q - \bar{Q}}{W} \right\|_\infty \int_{\mathbb{T}\times\mathbb{R}\times\mathbb{T}} W p \,\mathrm{d}x\mathrm{d}y\mathrm{d}\theta$$

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The Lyapunov bound

Note that $\mathcal{L} = y\partial_x - \kappa \sin x \partial_y + h(x, y, \theta)\partial_\theta + L_y$ where

$$L_y = -\gamma y \partial_y + \frac{\sigma^2}{2}\partial_{yy}$$

is the generator of an OU process with stationary density $e^{-\gamma y^2/\sigma^2}$.
So, we choose $W(x, y, \theta) = e^{\gamma y^2/(2\sigma^2)}$. Again $\mathcal{L}W$ is p-mean zero, but

$$\mathcal{L}W \leq -cW + d, \quad d = \sup_y \left[\frac{\gamma}{2\sigma^2}(-\gamma y^2 + \sigma^2 + 2\kappa|y|) + c\right] e^{\gamma y^2/(2\sigma^2)}$$

for all $c \in \mathbb{R}$.

$$\int_{\mathbb{T}\times\mathbb{R}\times\mathbb{T}} (\mathcal{L}W) p \mathrm{d}x \mathrm{d}y \mathrm{d}\theta = 0 \implies \int_{\mathbb{T}\times\mathbb{R}\times\mathbb{T}} W p \mathrm{d}x \mathrm{d}y \mathrm{d}\theta \leq \frac{d}{c}$$

W is called a **Lyapunov function** and is used in ergodicity proofs.

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$$L_y H_m(\sqrt{\gamma} y/\sigma) = -\gamma m H_m(\sqrt{\gamma} y/\sigma).$$

where the H_m's are **Hermite polynomials**. We use the Hilbert basis

$$\{f_n\} = \{\cos kx, \sin kx\} \otimes \{h_m(\sqrt{\gamma} y/\sigma)\} \otimes \{\cos n\theta, \sin n\theta\},$$

where we choose the normalisation $h_m = H_m/\sqrt{2^m m!}$. The suprema

$$\sup_{y \in \mathbb{R}} \left| \frac{h_m(\sqrt{\gamma} y/\sigma)}{W(x, y)} \right| = \sup_{z \in \mathbb{R}} \left| h_m(z) e^{-z^2/2} \right|$$

can be bounded by computer assistance or with a priori bounds.

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Result for the randomly forced pendulum

Theorem

Consider the noisy pendulum equation on $\mathcal{M} = \mathbb{T} \times \mathbb{R}$

$$\begin{cases} \mathrm{d}x_t &= y_t \mathrm{d}t \\ \mathrm{d}y_t &= -(\kappa \sin x_t + \gamma y_t)\mathrm{d}t + \sigma \mathrm{d}B_t \end{cases}$$

For $\kappa = 2/3, \gamma = 1/4$ and $\sigma = 4$, we have the following bounds for λ

$$\lambda \in [0.0228, 0.0315].$$

$$\mathcal{L} = y\partial_x - (\kappa \sin x + \gamma y)\partial_y + h(x, y, \theta)\partial_\theta + \frac{\sigma^2}{2}\partial_{yy}$$

is very much **non-elliptic** (in both the directions x and θ) and even accounting for the symmetries of the system, for this result, we had to use $105, 568, 462$ basis functions!
Finding $\bar{u}$ was the difficult task here.

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The Hopf system

Now, on $\mathbb{R}^2$, consider the Hopf normal form with additive noise

$$\mathrm{d}\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \left[\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix} - \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix}(x_t^2 + y_t^2)\right]\mathrm{d}t + \sigma \mathrm{d}B_t.$$

Recent results show that the Lyapunov exponent λ_b becomes positive as $b \to \infty$ (Chemnitz & Engel (CMP 2023), Baxendale (PTRF 2024)). We aim to show this result for a finite value b.
One can show that

$$\lambda_b = \int_{\mathbb{R}_+ \times \mathbb{T}} Q(r,\psi)p(r,\psi)\mathrm{d}r\mathrm{d}\psi, \quad Q(r,\psi) = \alpha - 2ar^2 + \sqrt{a^2+b^2}\, r^2 \sin 2\psi$$

and p is the ergodic distribution of a diffusion with generator

$$\mathcal{L}_b = (\alpha r - ar^3)\frac{\partial}{\partial r} + \left(br^2 + \sqrt{a^2+b^2}\, r^2 \cos 2\psi\right)\frac{\partial}{\partial \psi} + \frac{\sigma^2}{2}\Delta$$

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We aim to enclose λ_b on $[0, 30]$ by constructing

$$\bar{u}_b(r,\psi) = \sum_{k=1}^{K}\sum_{n} c_{k,n} T_k(b) f_n(r,\psi),$$

where the T_k's are Chebyshev polynomials.
The tricky bit is to handle the term $\sqrt{a^2+b^2}$ as it is non-polynomial. It is approximated by a polynomial $q(b)$ such that

$$\sup_{b\in[0,30]} |\sqrt{a^2+b^2} - q(b)| \le \varepsilon$$

This allows to quantitatively control the approximation

$$\hat{\mathcal{L}}_b = (\alpha r - ar^3)\frac{\partial}{\partial r} + \left(br^2 + q(b)r^2\cos 2\psi\right)\frac{\partial}{\partial \psi} + \frac{\sigma^2}{2}\Delta \approx \mathcal{L}_b$$

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A general setting

- We have a Hilbert (orthonormal) basis $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}^\infty(\mathcal{M}\times\mathbb{S}^{d-1})$ for a judicious Hilbert space $(\mathcal{H},(\cdot,\cdot))$ such that $f_0 = 1$.
- For all $N\in\mathbb{N}$, there exists $M\geq N$ such that $\mathcal{L}$ maps $P_N(\mathcal{H}) = \mathrm{Span}\{f_n\}_{n=0}^N$ to $P_M(\mathcal{H}) = \mathrm{Span}\{f_m\}_{m=0}^M$, where P_n denotes the projection in $\mathcal{H}$ onto $\mathrm{Span}\{f_i\}_{i=0}^n$.
- There is $n_0\in\mathbb{N}$ such that $Q\in\mathrm{Span}\{f_n\}_{n=0}^{n_0}$.
- There is a $\mathcal{C}^\infty$ weight function $W : \mathcal{M}\times\mathbb{S}^{d-1}\to\mathbb{R}_+$ such that

$$\sup_{m\in\mathbb{N}} \|f_m/W\|_\infty < \infty,$$

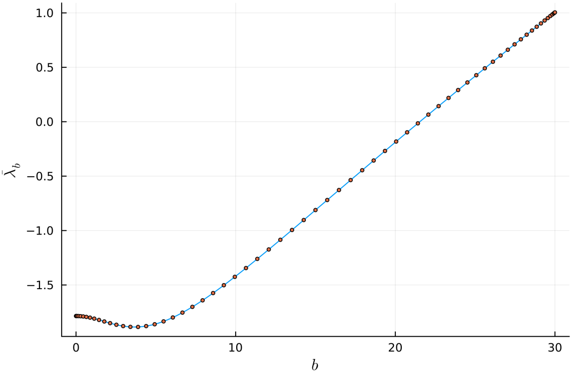
$$\text{and} \int_{\mathcal{M}\times\mathbb{S}^{d-1}} Wp\,\mathrm{d}x\,\mathrm{d}s < \infty$$

  which can be bounded explicitly.

We can choose $\bar{u} \approx \underset{u\in P_N(\mathcal{H})}{\operatorname{argmin}} \|P_0^\perp(\mathcal{L}u - Q)\|_{\mathcal{H}}^2$ and

$$\bar{\lambda} \stackrel{\text{def}}{=} P_0(Q-\mathcal{L}\bar{u}) \qquad \text{and} \qquad \bar{Q} \stackrel{\text{def}}{=} \mathcal{L}\bar{u} + \bar{\lambda} \approx Q,$$

Results



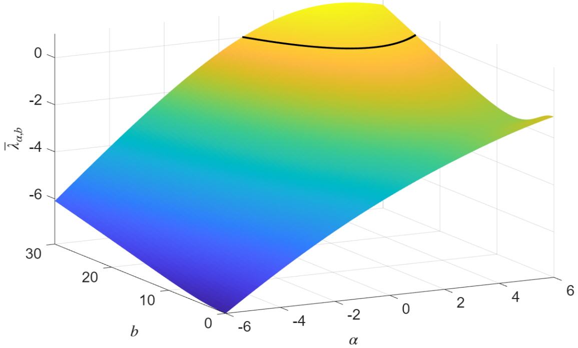
Theorem

Consider the Hopf normal form with $a = \alpha = 4$, $\sigma = \sqrt{2}$ and let $b \mapsto \bar{\lambda}_b$ be the function represented above. Then, for **all** $b \in [0, 30]$

$$|\lambda_b - \bar{\lambda}_b| \leq 3.41 \times 10^{-4}.$$

2D continuation

$$\mathrm{d}\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \left[\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix} - \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix}(x_t^2 + y_t^2)\right]\mathrm{d}t + \sigma \mathrm{d}B_t.$$



Theorem

If $a = 4$ and $\sigma = \sqrt{2}$, for all $(\alpha, b) \in [-6, 6] \times [0, 30]$,

$$|\lambda_{\alpha,b} - \bar{\lambda}_{\alpha,b}| \leq 4 \times 10^{-3}.$$

Thank you!

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