Arnold Diffusion and Blenders in the Three-Body Problem

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AGH University of Kraków

Joint work with

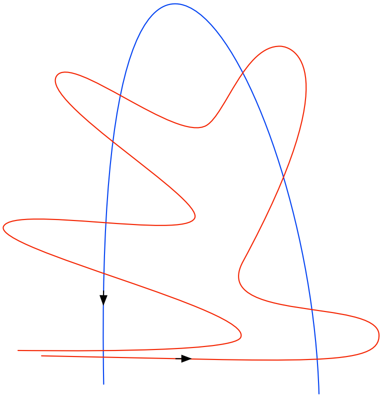
M. Gidea, W. Tucker, P. Zgliczyński

Plan of the presentation

- Hyperbolic fixed point $\rightarrow$ horseshoe
- Including a center direction
- Horizontal discs and existence of blenders
- Hyperbolicity from "twist and tilt"
- Planar three body problem

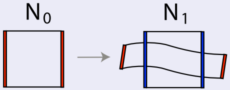
Symbolic dynamics

$$f : \mathbb{R}^n \to \mathbb{R}^n$$



Definition (covering relation)

We say that $N_0 \overset{f}{\Longrightarrow} N_1$ iff



$$M_0 \overset{f}{\Rightarrow} N_{i_1} \overset{f}{\Rightarrow} N_{i_k} \overset{f}{\Rightarrow} M$$

$$M_1 \overset{f}{\Rightarrow} N_{j_1} \overset{f}{\Rightarrow} N_{j_m} \overset{f}{\Rightarrow} M$$

[ZG] Zgliczyński, Gidea, Covering relations for multidimensional dynamical systems, JDE 2004

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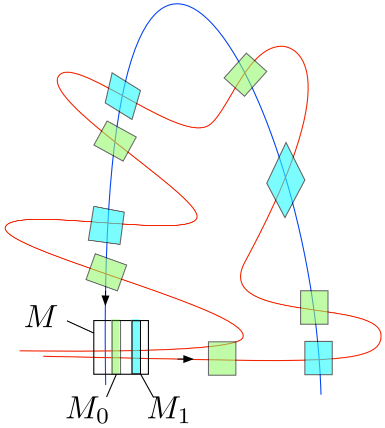

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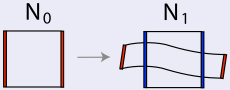

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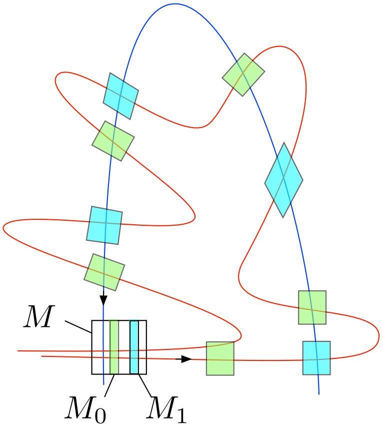

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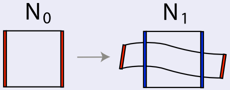


For $\ell, \kappa \in \{0, 1\}$

$$M_\ell \overset{f}{\Rightarrow} \cdots \overset{f}{\Rightarrow} M_\kappa$$

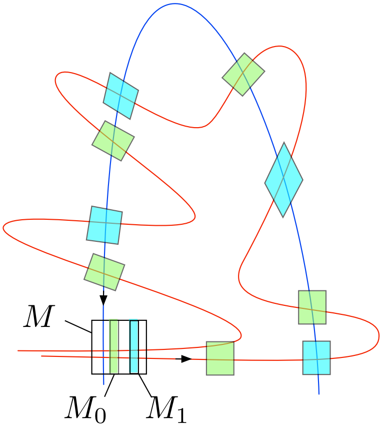
We have symbolic dynamics:

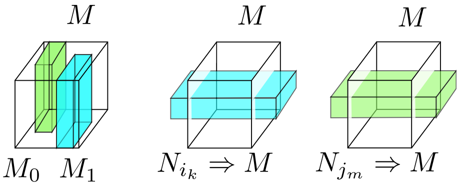
green / blue

[ZG] Zgliczyński, Gidea, Covering relations for multidimensional dynamical systems, JDE 2004

Including a ‘center’ coordinate

$f : \mathbb{R}^n \to \mathbb{R}^n$





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There is no way to continue... ☹

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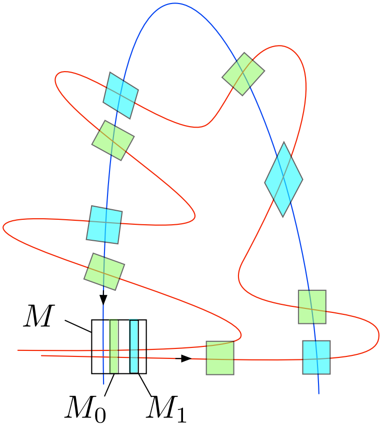
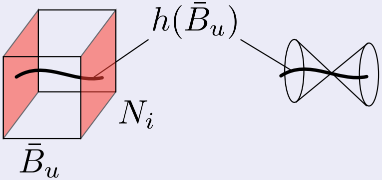


$N_i = \bar{B}_u \times \bar{B}_s$

Definition (horizontal disc h in N_i)

$h : \bar{B}_u \to N_i$, with cone aligned graph



Theorem

h_0 horizontal disc in N_0.
If $N_0 \Rightarrow N_1$ and f satisfies cone conditions, then
$h_1 = f(h_0) \cap N_1$ horizontal disc in N_1.

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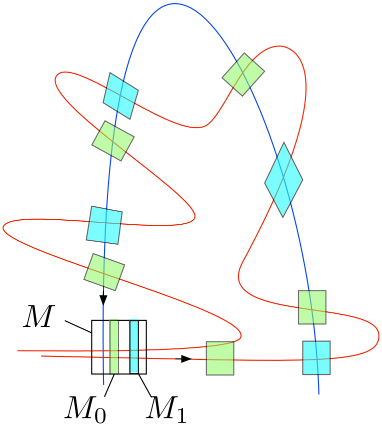

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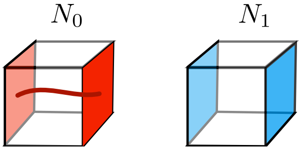

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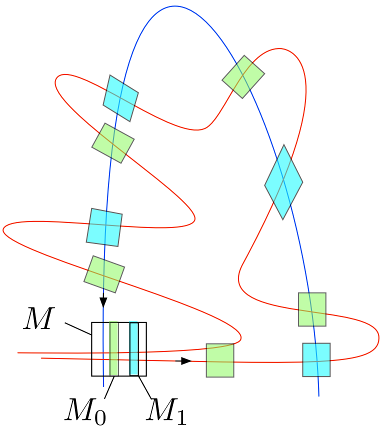

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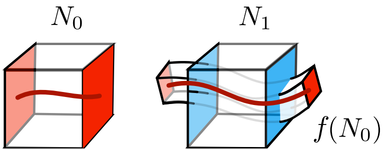

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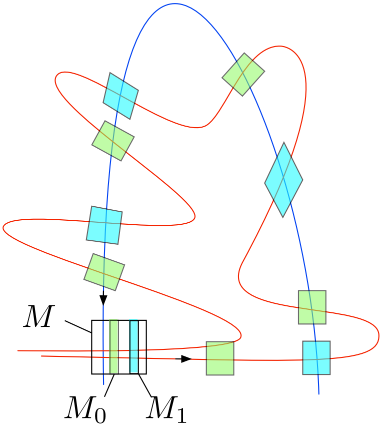

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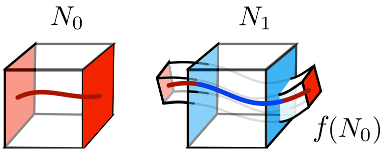

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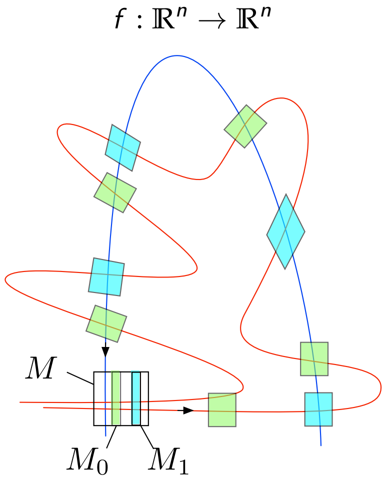
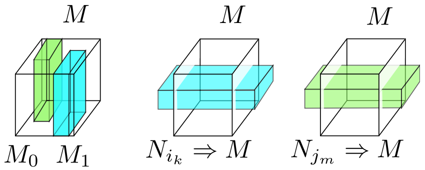
Forward invariant set and blenders





Theorem

Assume f satisfies c.c.
Assume $\forall h$ in M, $\exists \ell \in \{0,1\}$ s.t. $h \cap M_\ell$ is a horizontal disc in M_ℓ and

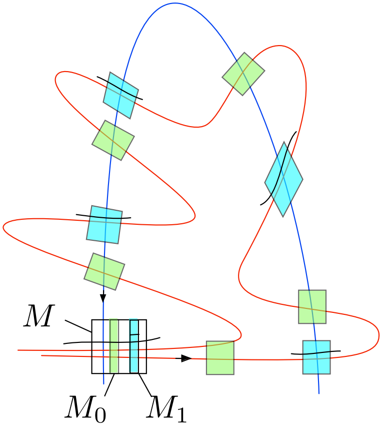
$$M_\ell \overset{f}{\Rightarrow} \cdots \overset{f}{\Rightarrow} M$$

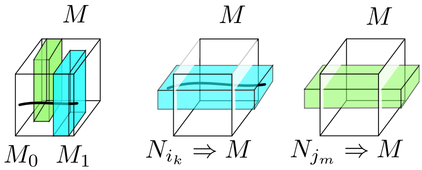
Then $\exists p \in M$ such that

$$f^n(p) \in \bigcup N_i \cup M$$

Forward invariant set and blenders

$f : \mathbb{R}^n \to \mathbb{R}^n$





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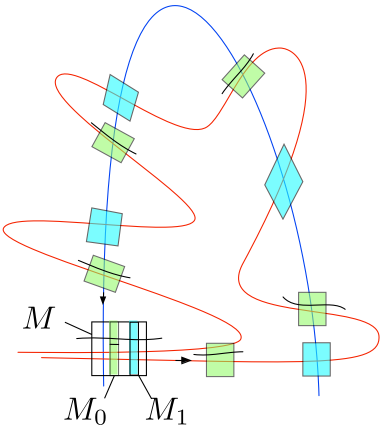

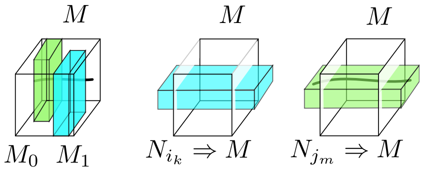

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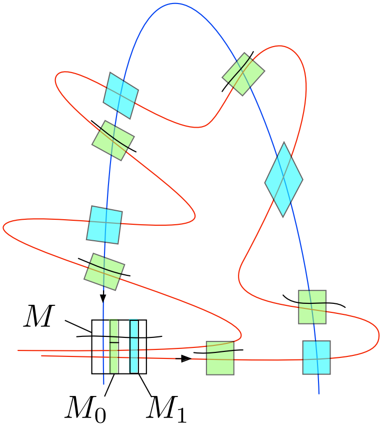
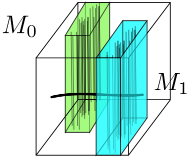


[CKOZ] MC B. Krauskopf, H. Osinga, P. Zgliczyński, JDE 2025



Theorem (existence of blenders)

Assume f satisfies c.c.
$\Lambda = inv(\bigcup N_i \cup M)$ *is hyperbolic.*
Assume $\forall h$ *in* M, $\exists \ell \in \{0,1\}$ *s.t.*
$h \cap M_\ell$ *is a horizontal disc in* M_ℓ *and*

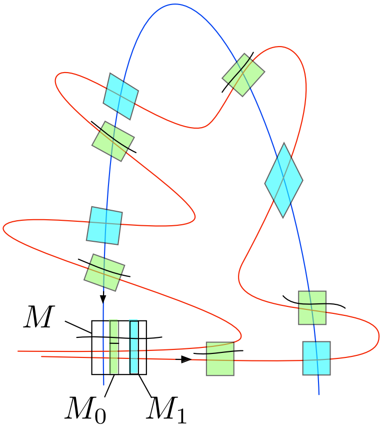
$$M_\ell \overset{f}{\Rightarrow} \cdots \overset{f}{\Rightarrow} M$$

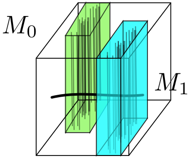
Also $dim(h) < dim W^u_\Lambda$. *Then*

$$h \cap W^s_\Lambda \neq \emptyset$$

Forward invariant set and blenders

$$f : \mathbb{R}^n \to \mathbb{R}^n$$





$$Q = diag(Id_u, -Id_s)$$

Theorem ([W])

For $p \in f^{-1}(N_j) \cap N_i$

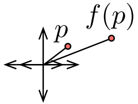
$$Df_{ji}(p)^T Q Df_{ji}(p) - Q$$

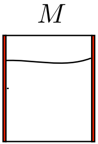
is strictly positive definite, then $inv(\bigcup N_i)$ is hyperbolic.

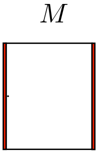
[W] D. Wilczak. Uniformly hyperbolic attractor of the Smale-Williams type ... SIADS 2010.

The main idea

Two connecting sequences of covering relations







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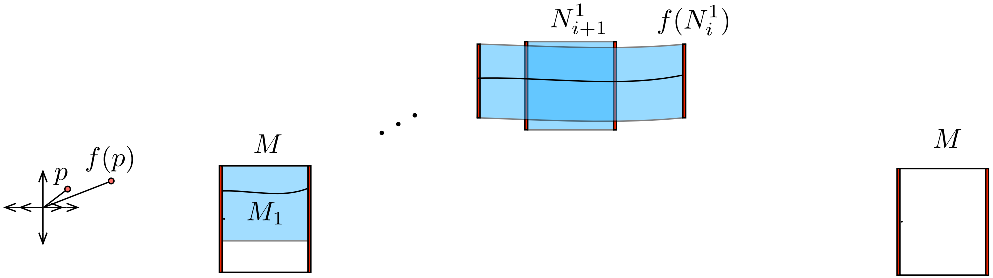

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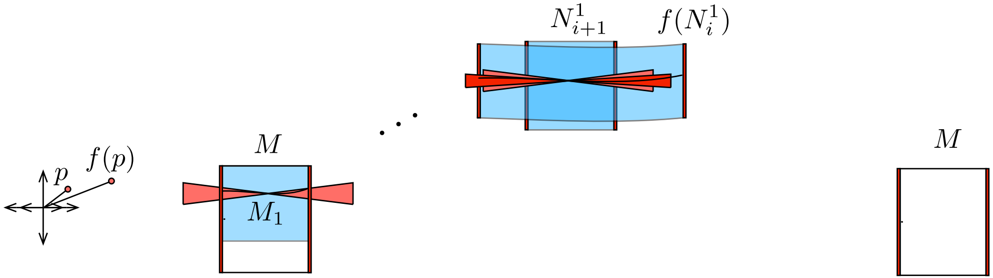

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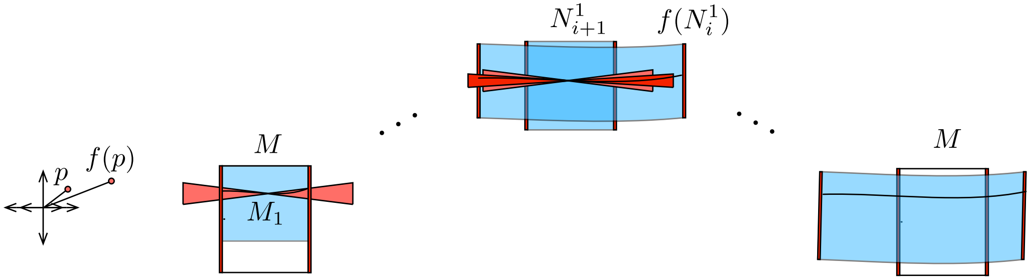

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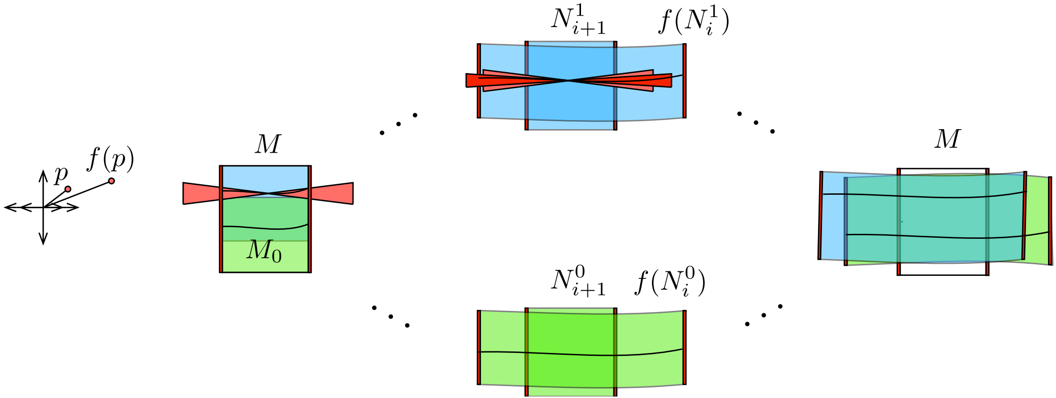
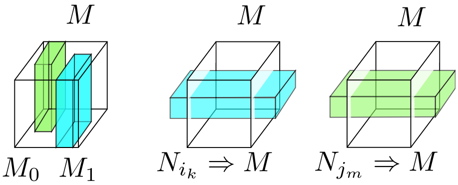
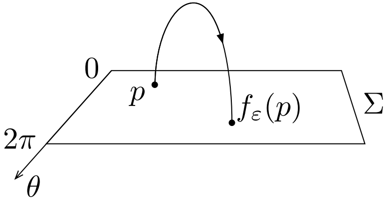


Hamiltonian systems





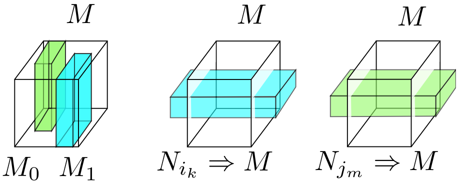
$$\mathbf{x}' = J\nabla_{\mathbf{x}}(H(\mathbf{x}) + \varepsilon G(\mathbf{x}, t))$$

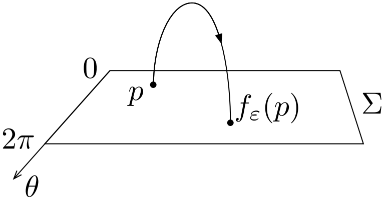
conservation of energy for $\varepsilon = 0$

$$H(\Phi_t^{\varepsilon=0}(\mathbf{x})) = H(\mathbf{x})$$

$$p = (x, y, \theta, I)$$
$$I = H(p)$$
$$I(f_{\varepsilon=0}(p)) = I(p)$$

Hamiltonian systems





$$\mathbf{x}' = J\nabla_{\mathbf{x}}(H(\mathbf{x}) + \varepsilon G(\mathbf{x}, \theta))$$
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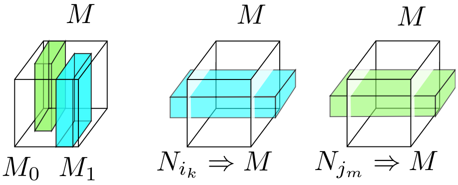

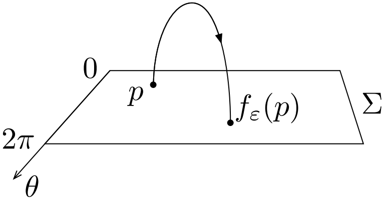

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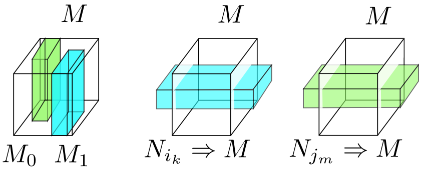

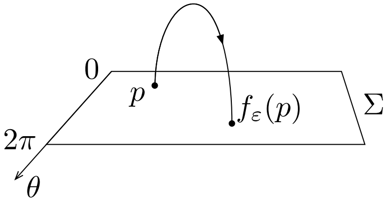

Theorem

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Assume $\forall h$ in M, $\exists \ell \in L$ s.t.
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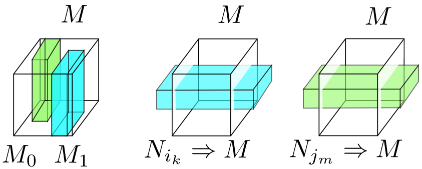

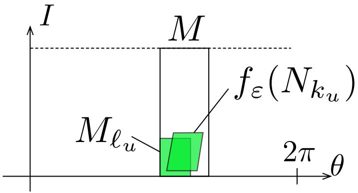

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Controlling the energy change:

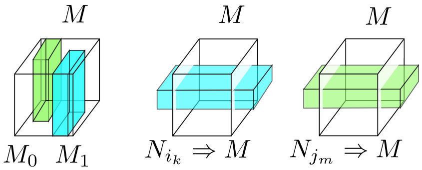
for h in M, $\exists \ell_u \in L$ s.t.

$$M_{\ell_u} \overset{f_\varepsilon}{\Rightarrow} \cdots \overset{f_\varepsilon}{\Rightarrow} N_{k_u} \overset{f_\varepsilon}{\Rightarrow} M$$

$$I(f_\varepsilon^{k_u+1}(p)) - I(p) > c\varepsilon$$

Arnold diffusion

Hamiltonian systems



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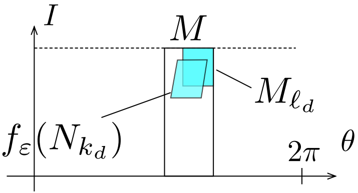


$$p = (x, y, \theta, I)$$

Controlling the energy change.

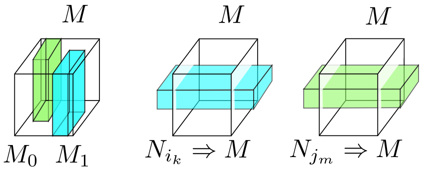
for h in M, $\exists \ell_d \in L$ s.t.

$$M_{\ell_d} \overset{f_\varepsilon}{\Rightarrow} \cdots \overset{f_\varepsilon}{\Rightarrow} N_{k_d} \overset{f_\varepsilon}{\Rightarrow} M$$

$$I(f_\varepsilon^{k_d+1}(p)) - I(p) < -c\varepsilon$$

Arnold diffusion

Hamiltonian systems



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[CG1] MC M. Gidea "Arnold diffusion ... 3BP" CPAM 2023

[CG2] MC M. Gidea "Arnold diffusion in the full 3BP" Arxiv

$$p = (x, y, \theta, I)$$

Controlling the energy change.

for h in M, $\quad \exists \ell_d \in L$ s.t.

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Arnold diffusion

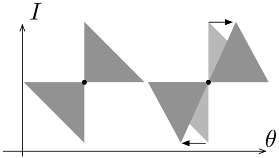
Hyperbolicity for blenders

$(x, y, \theta, I) \in \Sigma$

- x, y are hyperbolic coordinates,
- θ, I are center coordinates,
- **I constant of motion if $\varepsilon = 0$**

Two model maps:

$$(\theta, I) \mapsto (\theta + \omega I, I)$$



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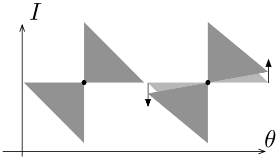

Hyperbolicity for blenders

$(x, y, \theta, I) \in \Sigma$

- x, y are hyperbolic coordinates,
- θ, I are center coordinates,
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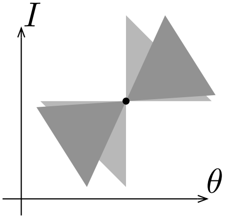

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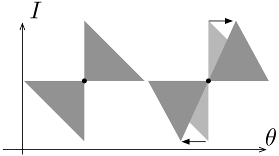

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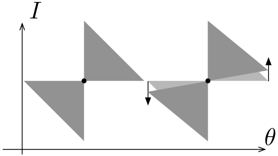

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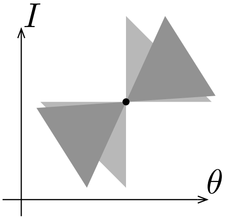

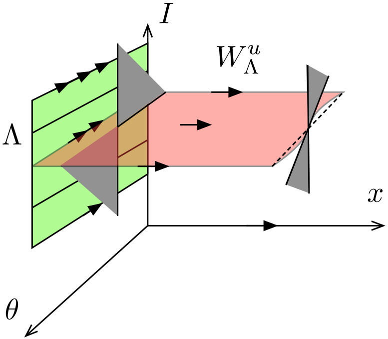

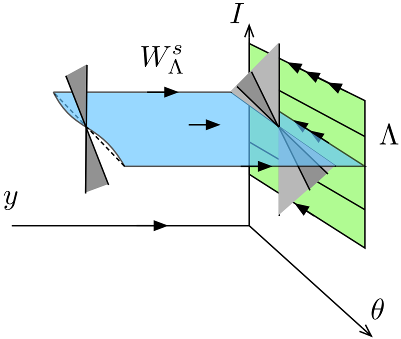

Hyperbolicity for blenders

hyperbolicity from perturbation

$$(\theta, I) \mapsto (\theta + \omega I, I)$$
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$$f_\varepsilon = \begin{pmatrix} 1+\varepsilon\beta\omega & \omega \\ \varepsilon\beta & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{Cone} = \{v \in \mathbb{R}^2 : v^T Q v \geq 0\}$$

Not good:

$$Df_\varepsilon^T Q Df_\varepsilon - Q = \begin{pmatrix} 0 & 0 \\ 0 & 2\omega \end{pmatrix} + O(\varepsilon)$$

Trick/idea:

$$A_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\varepsilon} \end{pmatrix} \qquad \tilde{f}_\varepsilon = A_\varepsilon^{-1} f_\varepsilon A_\varepsilon$$

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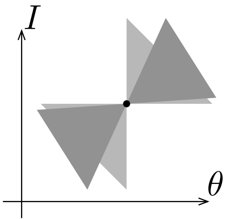

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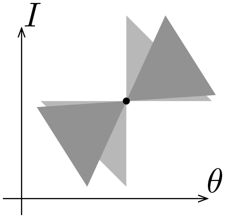

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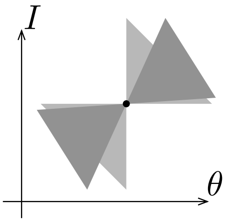

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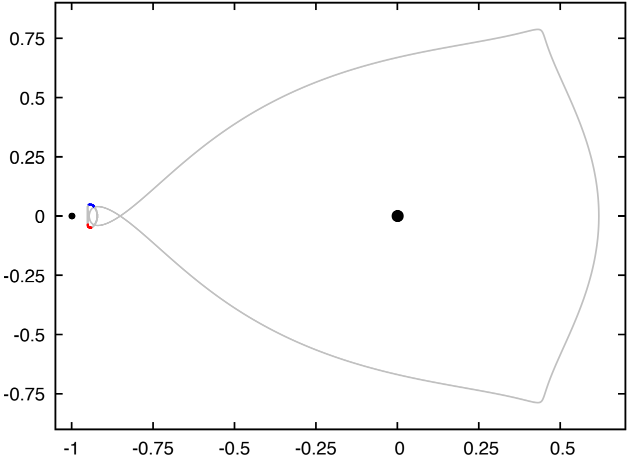
Blenders in the planar 3bp

$$\mathbf{x}' = J\nabla_{\mathbf{x}}(H(\mathbf{x}) + \varepsilon G(\mathbf{x}, t))$$



Work in progress with M. Gidea, W. Tucker and P. Zgliczyński

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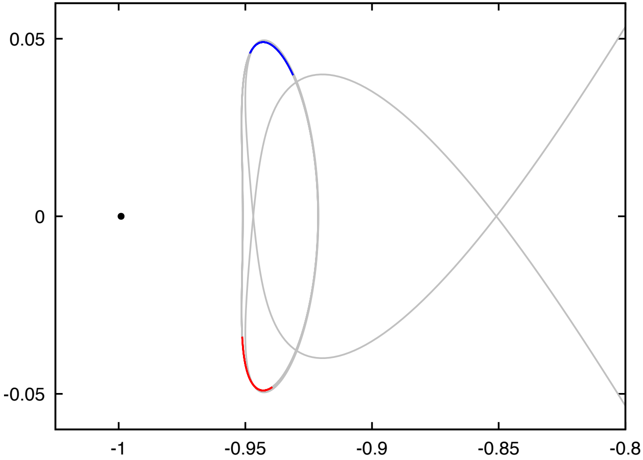

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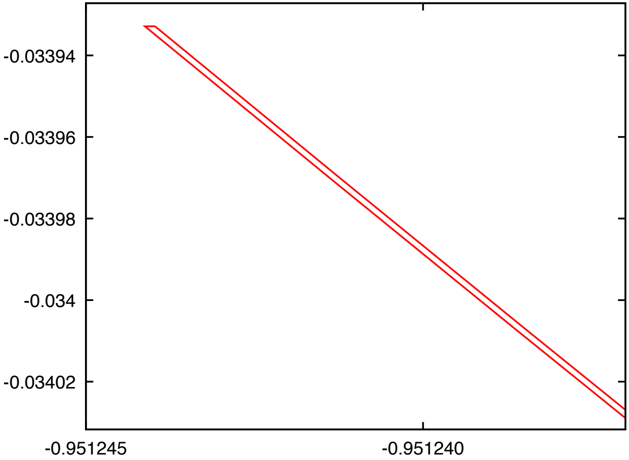
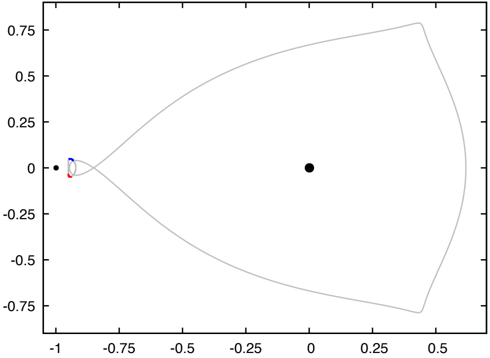


Work in progress with M. Gidea, W. Tucker and P. Zgliczyński

Closing remarks

- We can have horseshoe–type dynamics even when coordinates do not cooperate
- Future work:
  - Dynamical consequences of blenders in celestial mechanics
  - Transitivity



Thank you for your attention