Solitary waves in the Whitham equation

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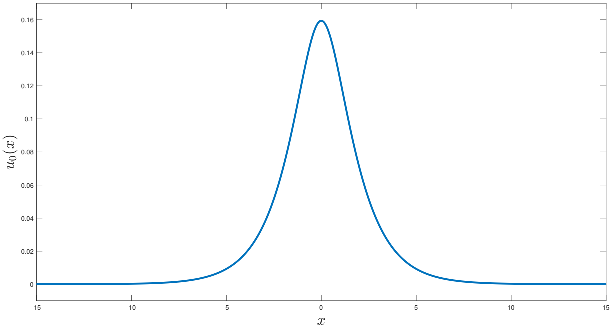

The Whitham equation is a nonlocal equation modeling the surface of 2D water waves

$$u_t + \partial_x \mathbb{M} u + \frac{1}{2} u \partial_x u = 0, \quad m(\xi) = \mathcal{F}(\mathbb{M})(\xi) = \sqrt{\frac{\tanh(2\pi\xi)}{2\pi\xi}} \tag{1}$$

It is a simplified water-wave model for wave breaking, peaking, using the full nonlinear dispersive relation m.

It generalizes the KdV equation, which is a simplification for small wavenumbers

$$\sqrt{\frac{\tanh(2\pi\xi)}{2\pi\xi}} \approx 1 - \frac{1}{6}(2\pi\xi)^2 + \mathcal{O}(\xi^4)$$

leads to

$$u_t + \partial_x u + \frac{1}{6}\partial_x^3 u + \frac{1}{2} u \partial_x u = 0. \tag{2}$$

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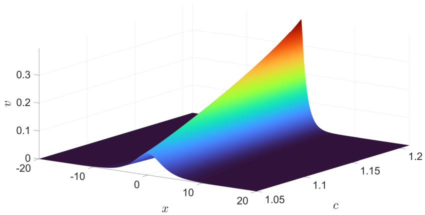
Solitary wave solutions

Looking for solitary (traveling) wave solutions $u = u(x - ct)$ vanishing at infinity, the equation becomes

$$\mathbb{M}u - cu + u^2 = 0, \quad \mathcal{F}(\mathbb{M})(\xi) = \sqrt{\frac{\tanh(2\pi\xi)}{2\pi\xi}} = m(\xi), \quad \lim_{|x|\to\infty} u = 0. \tag{3}$$



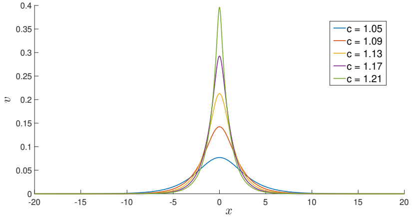


Figure: Branch of solitary waves parameterized by c

Localized solutions exists only for $c > 1$. In particular

$$m(\xi) - c < 0 \quad \text{if } c > 1 \text{ so} \quad \mathbb{M} - cI : L^2 \to L^2 \text{ is invertible.} \tag{4}$$

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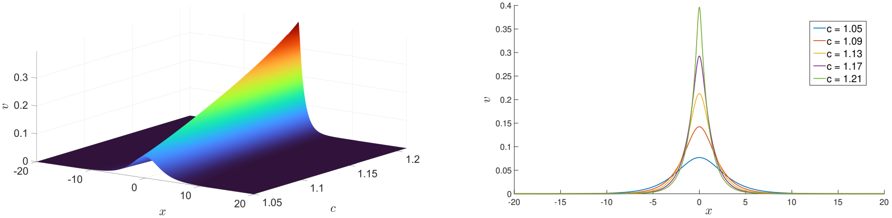

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Known results: existence of solitary waves

Traveling wave problem: find $u : \mathbb{R} \to \mathbb{R}$ vanishing at infinity with

$$\mathbb{M}u - cu + u^2 = 0, \quad \mathcal{F}(\mathbb{M})(\xi) = \sqrt{\frac{\tanh(2\pi\xi)}{2\pi\xi}} = m(\xi).$$

Variational methods (concentration-compactness):

- Existence and *conditional energetic stability* in a broad class of nonlocal equations (Albert; Arnesen)
- Family of *small-amplitude* waves in the Whitham equation, approximated by KdV solitons (Ehrnström, Groves and Wahlén).

Bifurcation / perturbation from KdV (for $c \approx 1$):

- Small-amplitude waves for c slightly above 1, with *spectral stability* (Stefanov and Wright)
- Bifurcation via the center manifold theorem (Johnson, Truong and Wheeler; Truong et al.).
- *Global* bifurcation and full families via period-limiting arguments (Arnesen et al.; Ehrnström, Nik and Walker; Truong et al.).

Highest / cusped waves:

- Proof of a highest, cusped *periodic* wave; extreme waves have $C^{1/2}$ regularity (Ehrnström and Wahlén).
- *Computer-assisted proof* of the convexity for the cusped periodic wave (Clausen functions capture the cusp) (Enciso, Gómez-Serrano and Vergara).

Stability:

- *Conditional energy* stability from concentration-compactness (Arnesen; Ehrnström, Groves and Wahlén).
- *Spectral* stability from perturbative KdV arguments, near $c \approx 1$ (Stefanov and Wright).

The numerical study

Since the equation is defined by a Fourier transform, solutions are looked for as Fourier series on large intervals $(-d, d)$

$$u(x) = \sum_{n=0}^{N} U_n \cos\left(\frac{n\pi}{d}\right).$$

Numerically we solve of $U = (U_n)$ satisfying

$$MU - cU + U * U = 0 \text{ and } (MU)_n = m\left(\frac{n}{2d}\right) U_n \text{ with } m(\xi) = \sqrt{\frac{\tanh(2\pi\xi)}{2\pi\xi}}.$$

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Constructive existence proof

From numerical solutions to $MU - cU + U * U = 0$, can we produce solutions to $\mathbb{M}u - cu + u^2 = 0$?

Given

$$\bar{u}(x) = \mathbf{1}_{(-d,d)}(x) \sum_{n=0}^{N} U_n \cos\left(\frac{n\pi}{d} x\right) \in H^2(\mathbb{R}),$$

prove that there exists $\tilde{u} \in H^2(\mathbb{R})$ such that

$$\mathbb{M}\tilde{u} - c\tilde{u} + \tilde{u}^2 = 0 \text{ and } \|\tilde{u} - \bar{u}\|_{H^2} \leq r \text{ with } r \text{ explicit.}$$

Reformulation of the problem

$$\mathbb{M}u - cu + u^2 = 0, \quad \mathcal{F}(\mathbb{M})(\xi) = \sqrt{\frac{\tanh(2\pi\xi)}{2\pi\xi}} = m(\xi).$$

Let

$$\nu \overset{\text{def}}{=} \frac{4}{\pi^2} \text{ and } \mathbb{L}_\nu \overset{\text{def}}{=} I - \nu\Delta. \tag{5}$$

We consider the zero finding problem

$$\mathbb{F}(u) \overset{\text{def}}{=} \mathbb{M}u - cu + u^2 = 0. \tag{6}$$

We define the Hilbert space H as

$$H \overset{\text{def}}{=} \overline{\{u \in L^2,\ u(x) = u(-x),\ \|u\|_H < \infty\}}^{\|\cdot\|_H} \text{ with } \|u\|_H \overset{\text{def}}{=} \|\mathbb{L}_\nu u\|_2 \tag{7}$$

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Proving that $\mathbb{F} : H \to L^2$ is well-defined

- $\mathbb{\Lambda}_\nu$ preserves the even symmetry

Lemma

The product $H \times H \to H$ is a bounded and $\|\mathbb{\Lambda}_\nu(uv)\|_2 \leq \kappa \|u\|_H \|v\|_H$ with $\kappa = \frac{\pi}{2}$

- $\mathbb{F} : H \to L^2$ is smooth (polynomial).

Regularity of solutions

Lemma

If $\tilde{u} \in H$ and $\mathbb{F}(\tilde{u}) = 0$ such that $2\tilde{u}(x) - c < 0$ for all $x \in \mathbb{R}$, then $\tilde{u} \in H^\infty(\mathbb{R})$.

Proof.

Taking a derivative

$$\mathbb{M}\tilde{u}' - c\tilde{u}' + 2\tilde{u}\tilde{u}' = 0 \tag{8}$$

and so

$$\tilde{u}' = (c - 2\tilde{u})^{-1}\mathbb{M}\tilde{u}' \in H^{\frac{3}{2}}(\mathbb{R})$$

using that $\tilde{u}' \in H^1$ and $\mathbb{M} : H^s \to H^{s+\frac{1}{2}}$. □

- Cusped wave : $2\tilde{u}(0) = c \implies$ only $C^{\frac{1}{2}}$ regularity

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Objectives

- Build a fixed point operator $\mathbb{T}(u) = u - D\mathbb{F}(\bar{u})^{-1}\mathbb{F}(u)$
  - Compute $\mathbb{F}(\bar{u})$
  - Invert $D\mathbb{F}(\bar{u})$
- Construction of branches of solutions
- Spectral stability : control of the spectrum

Applying a nonlocal operator

Recall that

$$\mathcal{F}(\mathbb{M})(\xi) = \sqrt{\frac{\tanh(2\pi\xi)}{2\pi\xi}} = m(\xi) \quad \text{and} \quad (MU)_n = m\left(\frac{n}{2d}\right) U_n \tag{9}$$

Let $u(x) = \mathbf{1}_{(-d,d)}(x) \sum_{n=0}^{N} U_n \cos\left(\frac{n\pi}{d} x\right) \in H^2$ and define

$$(Mu)(x) = \mathbf{1}_{(-d,d)}(x) \sum_{n=0}^{N} (MU)_n \cos\left(\frac{n\pi}{d} x\right). \tag{10}$$

Lemma

Let $v = (\frac{4}{\pi^2}\Delta - 1)u = \Lambda_\nu u$, then

$$\|\mathbb{M}u - Mu\|_{L^2} \le Ce^{-\frac{\pi}{2}d} \|ve^{\frac{\pi}{2}x}\|_{L^2} \tag{11}$$

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Proof.

Notice that

$$\mathbb{M}u = \mathbb{M}\mathbb{\Lambda}_\nu^{-1}v. \tag{12}$$

Then, we prove that

$$(\mathbb{M}(\Delta - \frac{4}{\pi^2})^{-1}v)(x) = \int_{\mathbb{R}} f(x-y)v(y)dx \text{ and } |f(x)| \leq Ce^{-\frac{\pi}{2}x}. \tag{13}$$

The above is a consequence that

$$l(\xi) = \sqrt{\frac{\tanh(2\pi\xi)}{2\pi\xi}}\frac{1}{(2\pi\xi)^2 + \frac{4}{\pi^2}} \tag{14}$$

is analytic on $\{z \in \mathbb{C},\ |Im(z)| < \frac{\pi}{2}\}$ and $l \in L^1(\mathbb{R})$. Then Paley-Wiener (Cauchy integral theorem). □

This allows to compute

$$\|\mathbb{F}(\bar{u})\|_2 \leq \|F(\bar{U})\|_2 + Ce^{-\frac{\pi}{2}d}\|ve^{\frac{\pi}{2}x}\|_{L^2}. \tag{15}$$

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Insight for inverting the Jacobian

Suppose that $2\tilde{u} - c < 0$ and let

$$K = \left(I - \frac{1}{c}\mathbb{M}\right)^{-1}\left((2\tilde{u} - c)^{-1} + \frac{1}{c}\right)\mathbb{M}. \quad (16)$$

Now observe

$$D\mathbb{F}(\tilde{u}) = (2\tilde{u} - c)\left(I - \frac{1}{c}\mathbb{M}\right)(I + K). \quad (17)$$

Lemma

$K : H \to H$ is compact. In particular $D\mathbb{F}(\tilde{u}) : H \to H$ is Fredholm.

- How to invert $2\tilde{u} - c$?

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Let $w(x) = \begin{cases} -\dfrac{1}{c} & \text{if } |x| > d \\ \displaystyle\sum_{n=0}^{N} W_n \cos\left(\frac{n\pi}{d}x\right) & \text{if } |x| \leq d \end{cases}.$

Let $\delta > 0$ *such that*

$$|1 - w(x)(2\bar{u}(x) - c)| \leq \delta \tag{18}$$

and suppose that $\|\tilde{u} - \bar{u}\|_{H^2} \leq r$. *If*

$$\epsilon = \delta + 2\kappa\|w\|_\infty r < 1, \tag{19}$$

then both $2\tilde{u} - c$ *and* $2\bar{u} - c$ *are invertible and*

$$\|(2\tilde{u} - c)^{-1} - w\|_\infty \leq \frac{\epsilon}{1-\epsilon} \quad \textit{and} \quad \|(2\bar{u} - c)^{-1} - w\|_\infty \leq \frac{\delta}{1-\delta}. \tag{20}$$

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Inverse of $D\mathbb{F}(\bar{u})$

In the goal of constructing a Newton fixed point operator

$$\mathbb{T}(u) = u - D\mathbb{F}(\bar{u})^{-1}\mathbb{F}(u), \tag{21}$$

with

$$D\mathbb{F}(\bar{u}) = (2\bar{u} - c)\left(I - \frac{1}{c}\mathbb{M}\right)(I + K), \quad K : H \to H \text{ compact.} \tag{22}$$

We construct a matrix B_N such that

$$I + B_N \approx (I + K)^{-1} \tag{23}$$

and define our approximate inverse as

$$\mathbb{A} = (I + B_N)\left(I - \frac{1}{c}\mathbb{M}\right)^{-1} w : H \to H. \tag{24}$$

With some heavy analysis, we show that

$$\|I - \mathbb{A}D\mathbb{F}(\bar{u})\|_H \lessapprox C_1\|1 - w(2\bar{u} - c)\|_\infty + C_2 e^{-\frac{\pi}{2}d}\|\bar{u}e^{\frac{\pi}{2}x}\|_{L^2} + \frac{C_3\sqrt{d}}{\sqrt{N}} \tag{25}$$

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We can compute

- $\|\mathbb{A}\mathbb{F}(\bar{\mathbb{u}})\|_H$
- $\mathbb{A}$ such that $\|I - \mathbb{A}D\mathbb{F}(\bar{u})\|_H < 1$

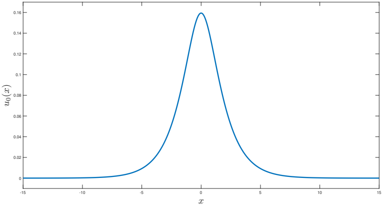
Using a Newton-Kantorovich argument, we get

Theorem (M.C. 2025)

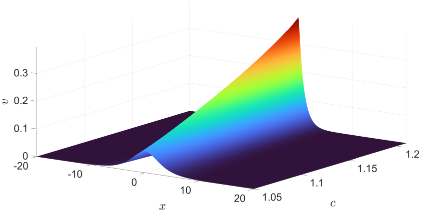
Let $r \stackrel{\text{def}}{=} 5.72 \times 10^{-9}$, then there exists a unique even solution $\tilde{u}$ to the WE in $\overline{B_r(\bar{u})} \subset H$ for $c = 1.1$. In addition, there exists a smooth curve

$$\{\tilde{u}(q) : q \in [d, \infty]\} \subset C^{\infty}(\mathbb{R})$$

such that $\tilde{u}(q)$ is a periodic solution to the WE with period $2q$. In particular, $\tilde{u}(\infty) = \tilde{u}$.



It has been proven (using global bifurcation) that solutions exists for all $c \in [1, c^*]$ where a cusped solution (peak) appears at $c = c^* \approx 1.23$.



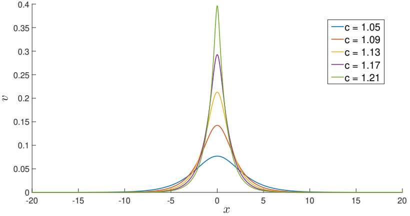


Figure: Branch of solitary waves parameterized by c

Can we quantify c^* precisely ? Convexity of the wave ? Possible branching out ?

Using the method developed by M. Breden and O. Hénot (to appear soon), we look for a branch of the form

$$\bar{u}(c) \stackrel{\text{def}}{=} \bar{u}_0 + 2\sum_{k=1}^{K} \bar{u}_k T_k\left(\frac{2(c-c_1)}{c_1-c_0}+1\right) \tag{26}$$

for all $c \in [c_0, c_1]$. Each $\bar{u}_k \in H$ is given as

$$\bar{u}_k(x) = \mathbf{1}_{(-d,d)}(x)\sum_{n=0}^{N}(\bar{U}_k)_n \cos\left(\frac{n\pi}{d}x\right). \tag{27}$$

- Construct $\bar{w}(c) = \bar{w}_0 + 2\sum_{k=1}^{K}\bar{w}_k T_k\left(\frac{2(c-c_1)}{c_1-c_0}+1\right) \approx (2\bar{u}(c)-c)^{-1}$
- Construct $\mathbb{A}(c) = (I + B_N(c))w(c) = \mathbb{A}_0 + 2\sum_{k=1}^{2K}\mathbb{A}_k T_k\left(\frac{2(c-c_1)}{c_1-c_0}+1\right)$
- Since all objects are polynomial in c, we can use the FFT to compute $\sup_c \|\mathbb{A}(c)\mathbb{F}(c)\|_H$ and $\sup_c \|I - \mathbb{A}(c)D\mathbb{F}(c)\|_H$

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Parameterization using Chebyshev

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Uniform contraction mapping theorem

Since $\mathbb{F}$ depends smoothly on c, the uniform contraction mapping theorem applies

Theorem

If $\mathbb{T}_c(u) = u - D\mathbb{F}(\bar{u}(c))^{-1}\mathbb{F}(u)$ is a contracting on $B_r(\bar{u}(c)) \to B_r(\bar{u}(c))$ for all $c \in [c_0, c_1]$, then there exists a smooth branch of solutions $\{\tilde{u}(c),\ c \in [c_0, c_1]\}$ such that $\|\tilde{u}(c) - \bar{u}(c)\|_H \leq r$.

Using the above, we prove the following result

Theorem (M.C. 2025)

For every $c \in [1.05, 1.21]$, there exists a smooth even solution $\tilde{u}(c) \in H^\infty(\mathbb{R})$ to the WE and the function $c \mapsto \tilde{u}(c)$ is smooth. Furthermore,

$$\sup_{c \in [1.05, 1.21]} \|\tilde{u}(c) - \bar{u}(c)\|_H \leq 3.2 \times 10^{-4}.$$

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For the stability of the traveling wave $\tilde{u}(x-ct)$ in time, we use

Lemma

Assume that
(P1) $D\mathbb{F}(\tilde{u})$ *has a simple positive eigenvalue* λ^+
(P2) $D\mathbb{F}(\tilde{u})$ *has no positive eigenvalue other than* λ^+
(P3) *0 is a simple eigenvalue of* $D\mathbb{F}(\tilde{u})$.
If

$$\left(\tilde{u}, D\mathbb{F}(\tilde{u})^{-1}\tilde{u}\right)_2 < 0, \tag{28}$$

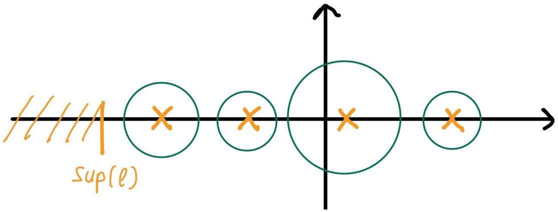
then $\tilde{u}$ *is (spectrally) stable.*

Consider the map

$$\mathbb{G}(\lambda, v) \stackrel{\text{def}}{=} \begin{pmatrix} (\bar{v} - v, \bar{v})_2 \\ D\mathbb{F}(\tilde{u})v - \lambda v \end{pmatrix} = 0 \tag{29}$$

where $\bar{v}$ is an approximate eigenfunction.

- Isolated zeros of $\mathbb{G} \implies$ simple eigenvalues
- $\|(D\mathbb{F}(\tilde{u}) - \lambda I)^{-1}\|_2 \leq C \implies$ no eigenvalues in $(\lambda - \frac{1}{C}, \lambda + \frac{1}{C})$.



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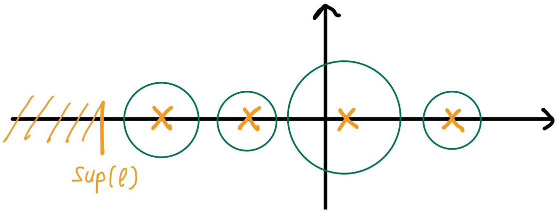

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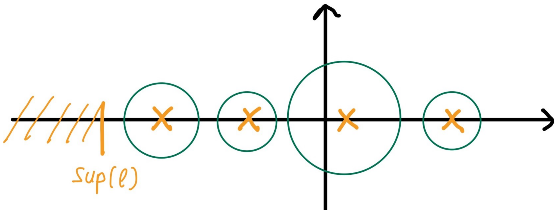

Theorem (M.C. 2025)

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Perspectives

- How to deal with the cusped solution ? (Clausen functions, see Gomez-Serrano et al.)
- Initial value problems
- Change the asymptotics at infinty (periodic to periodic, zero to periodic)