Rigorous computation of renormalization fixed points using Chebyshev series

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Joint work with Jorge Gonzalez and Jason Mireles James

ICMS Workshop - Validated Numerics for Computer-Assisted Proofs

Universality in period doubling cascades

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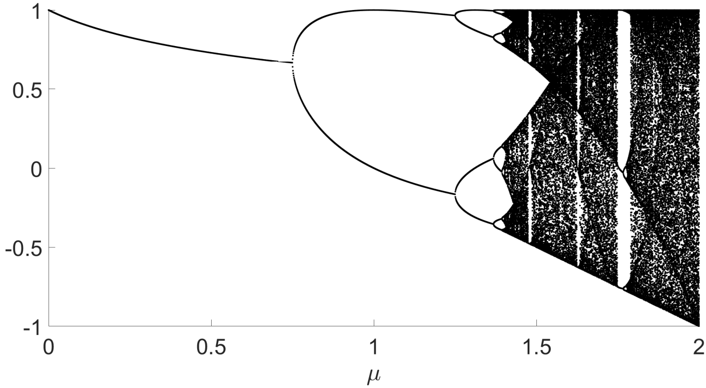

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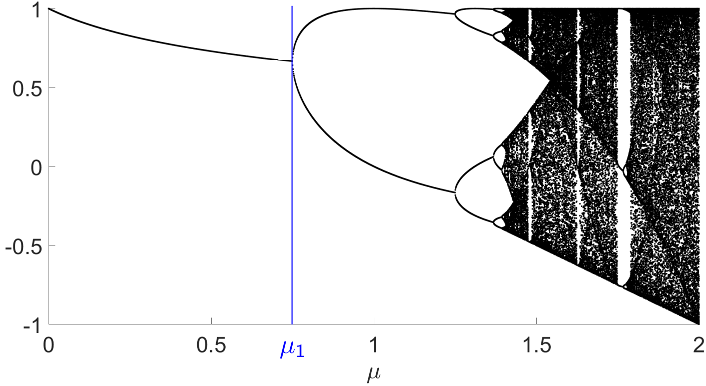

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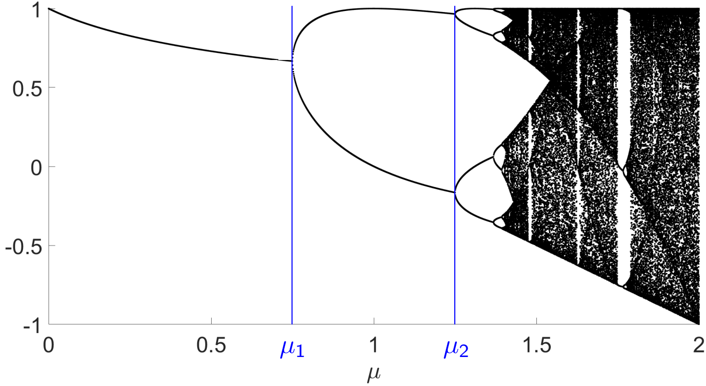

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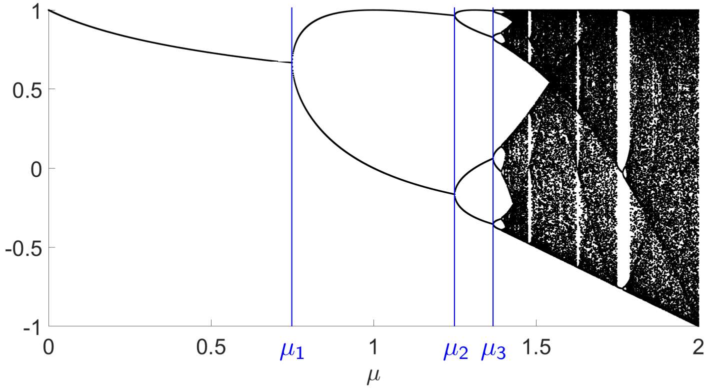

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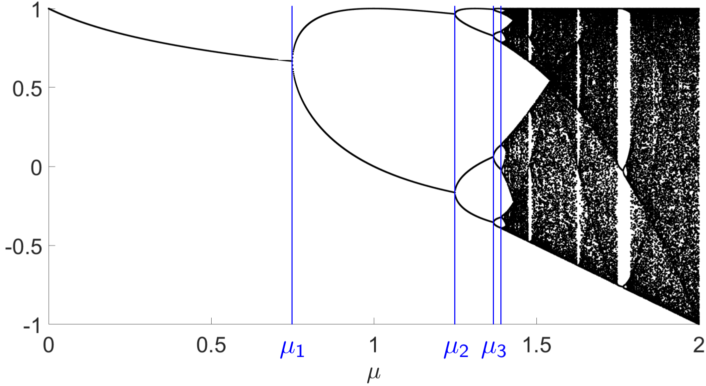

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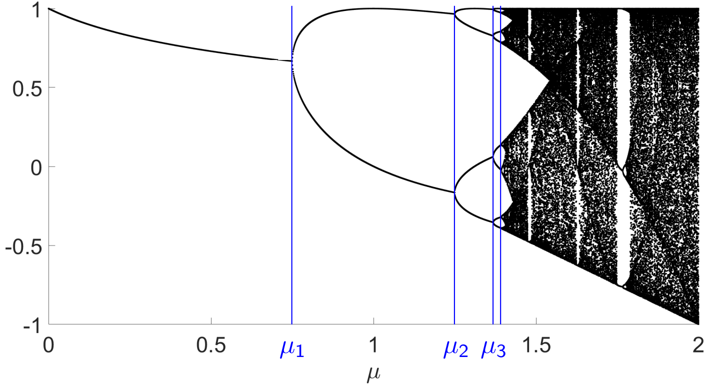

- The asymptotic ratio $\delta^\star = \lim_{k\to\infty} \dfrac{\mu_{k+1} - \mu_k}{\mu_{k+2} - \mu_{k+1}} \approx 4.67$ seems universal![1]

[1] Coullet and Tresser 1978; Feigenbaum 1978.

The Feigenbaum conjectures

- This universality can be explained by the existence of an hyperbolic fixed point $f^\star$ of the renormalization operator R_2 which acts on analytic even unimodal maps $f : [-1,1] \to [-1,1]$ as

$$R_2(f)(x) = \frac{1}{\alpha_2} f(f(\alpha_2 x)), \qquad \alpha_2 = f(f(0)),$$

provided $f^\star$ behaves quadratically near 0, $DR_2(f^\star)$ has a single unstable eigenvalue, and the unstable manifold has some geometric properties.[2] The universal constant $\delta^\star$ is then the unstable eigenvalue.

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- The Feigenbaum conjectures were first proven using early computer-assisted proofs.[3]

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  - Write $f(x) = 1 - x^2 h(x^2)$ and solve for h.

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- Consider the zero-finding problem $F(f) = f - R(f)$.
- Consider the norm $a|h_0| + \sum_{n\geq 1} |h_n|\rho^n$.
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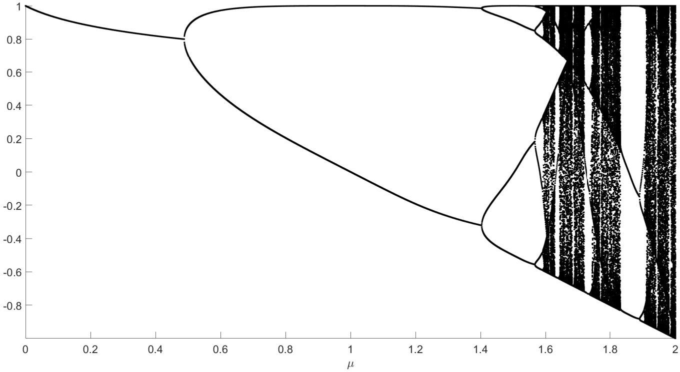
  - $\|T(\bar{f}) - \bar{f}\| \leq 4 \cdot 10^{-6}$,
  - $\|DT(f)\| \leq 0.9$ for all $\|f - \bar{f}\| \leq 0.01$.

Related problems (different degree)

Consider the sequence given by the quartic map: $x_{n+1} = 1 - \mu x_n^4$.



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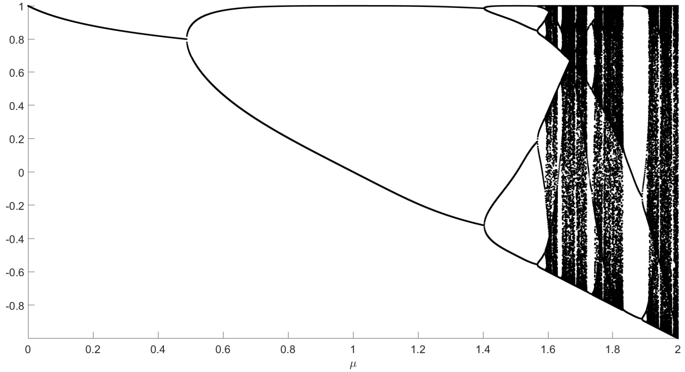

- This period-doubling cascade is described by a different universal constant.

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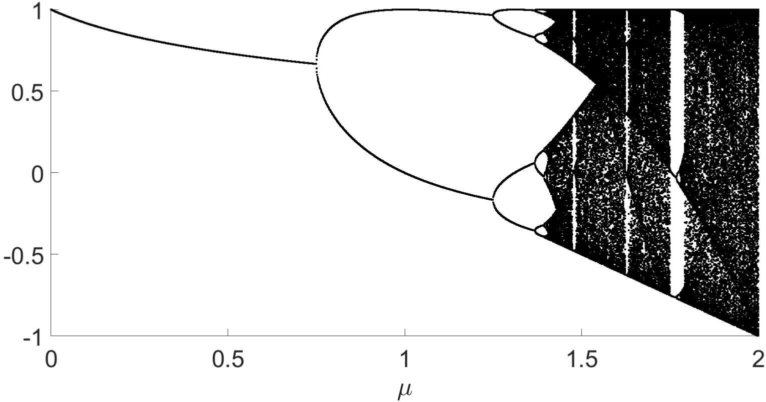

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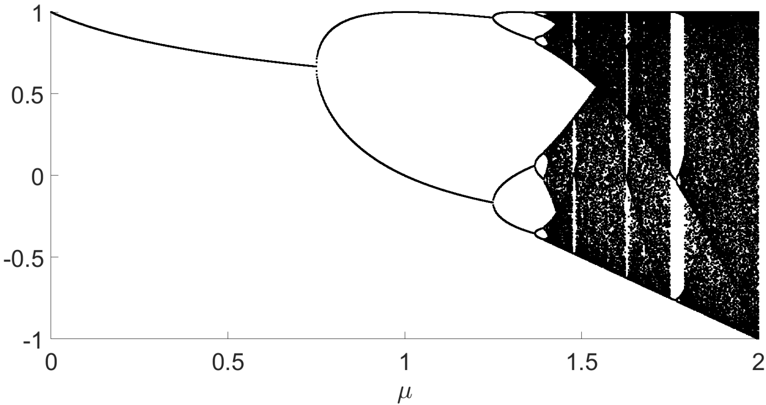

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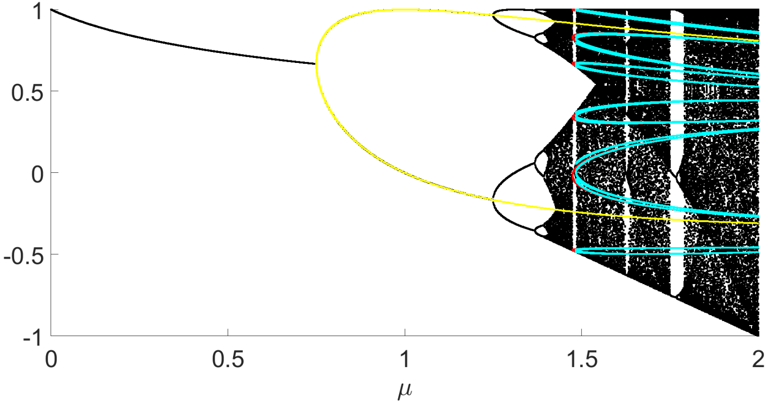

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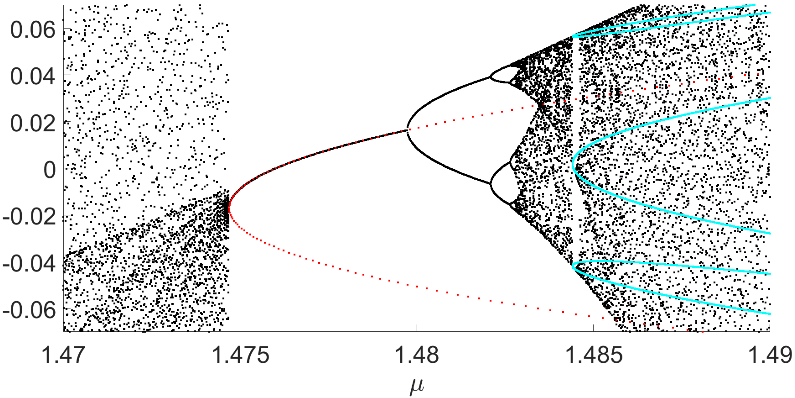

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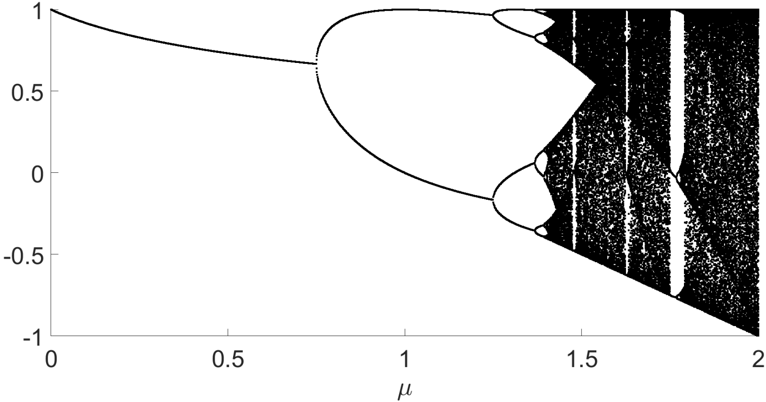

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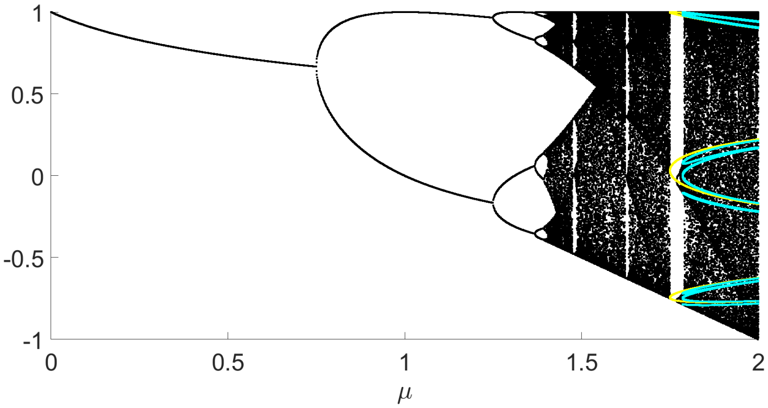

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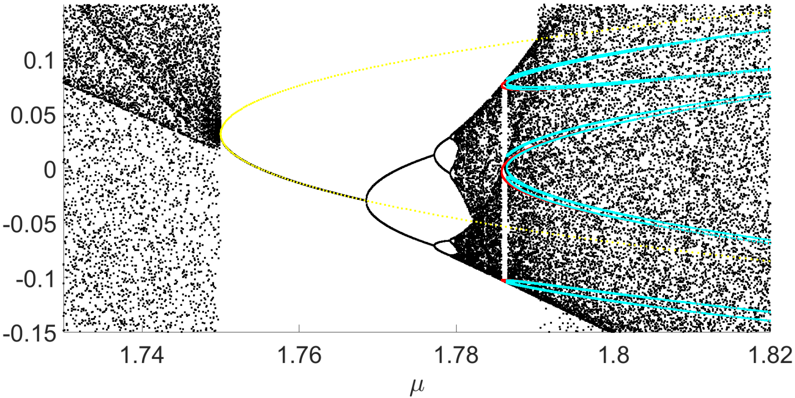

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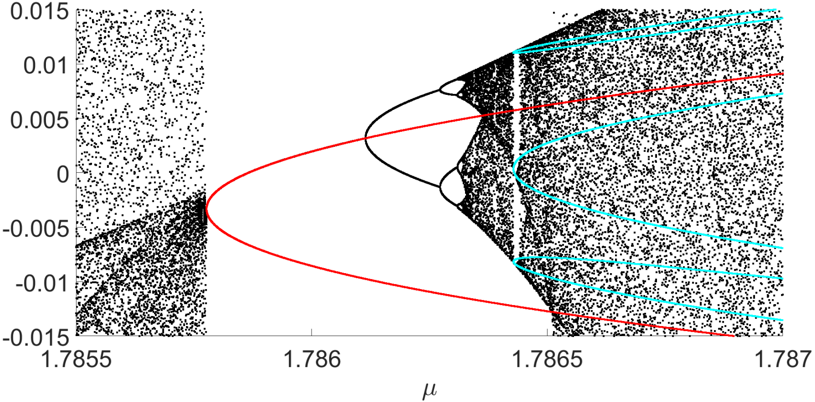

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- **Our goal:** develop a CAP methodology to study quantitatively all these fixed points and the associated universal constants.

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and the projection P_N is interpolation at $N+1$ Chebyshev nodes.

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  - For a computable h, like $h=\bar{\alpha}\bar{f}-\bar{f}^{(m)}(\alpha\cdot)$, this can be obtained using interval arithmetic, adaptive subdivision and Taylor expansions.

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- For α, f and h such that $|\alpha - \alpha^*| \leq r^*$, $\|f - \bar{f}\|_{\ell^1_\rho} \leq r^*$ and $\|h\|_{\ell^1_\rho} \leq 1$, we need to estimate quantities like

$$\left\| P_N \left[g_j(\cdot)(I - P_N) \left[h\left(f^{(j)}(\alpha \cdot) \right) \right] \right] \right\|_{\ell^1_\rho}, \quad g_j(x) = \prod_{l=j+1}^{m-1} f'\left(f^{(l)}(\alpha x) \right).$$

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- **Option 1:** evaluate at each Chebyshev node x_n, and pick $1 \leq \beta < \rho$ such that $f^{(j)}(\alpha x_n) \in \mathcal{E}_\beta$ for $0 \leq n \leq N$:

$$\begin{aligned} \left| g_j(x_n)(I - P_N) \left[h\left(f^{(j)}(\alpha x_n) \right) \right] \right| &\leq |g_j(x_n)| \, \|(I - P_N) h\|_{\mathcal{C}^0_\beta} \\ &\leq |g_j(x_n)| \, \tilde{C}_{\beta,\rho} \, \|h\|_{\ell^1_\rho}. \end{aligned}$$

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- **Option 2:**

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- **Option 2:** pick $1 \leq \beta < \rho < \gamma$ such that $f^{(j)}(\alpha \mathcal{E}_\gamma) \subset \mathcal{E}_\beta$, and estimate

$$\begin{aligned} \left\| g_j(\cdot)(I - P_N) \left[h \left(f^{(j)}(\alpha \cdot) \right) \right] \right\|_{\mathcal{C}^0_\gamma} &\leq \|g_j\|_{\mathcal{C}^0_\gamma} \|(I - P_N) h\|_{\mathcal{C}^0_\beta} \\ &\leq \|g_j\|_{\mathcal{C}^0_\gamma} \tilde{C}_{\beta,\rho} \|h\|_{\ell^1_\rho}, \end{aligned}$$

which gives a bound $\sim \gamma^{-n}$ for the Chebyshev coefficients.

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- Combining these, and taking $\nu > \rho$, one can easily get estimates like

$$\|(I - P_N)h\|_{\ell^1_\rho} \leq C_{\rho,\nu}(N)\,\|h\|_{\mathcal{C}^0_\nu} \quad \text{and} \quad \|(I - P_N)h\|_{\mathcal{C}^0_\rho} \leq \tilde{C}_{\rho,\nu}(N)\,\|h\|_{\ell^1_\nu}\,.$$

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- Analytic-type interpolation error estimates, with some back and forth between ℓ^1_ρ norms and $\mathcal{C}^0_\nu$ norms.
  - The rescaling factor $|\alpha| < 1$ gives us some room to play with.
- Rigorous enclosure of the supremum of (high degree) polynomials on the boundary of Bernstein ellipses.
- Combine everything into a Newton-like fixed point argument as usual.
  - We directly control $\|I - ADF(\alpha, f)\|$ for all (α, f) in a small ball of radius $r^\star$ around $(\bar{\alpha}, \bar{f})$, instead of splitting into

$$\|I - ADF(\bar{\alpha}, \bar{f})\| + \|A\left(DF(\bar{\alpha}, \bar{f}) - DF(\alpha, f)\right)\|.$$

  - Avoids a messy Z_2-type estimate, but requires taking r^* really small, and thus the usage of extended precision.
- Exact same strategy for enclosing the positive eigenvalue of $DR_m(f^\star)$, with

$$F(\lambda, u) = \begin{pmatrix} N(u) - 1 \\ DR_m(f^\star)u - \lambda u \end{pmatrix}.$$

Results

$$R_m(f)(x) = \frac{1}{\alpha_m} f^{(m)}(\alpha_m x), \qquad \alpha_m = f^{(m)}(0), \qquad DR_m(f)u = \delta_m u$$

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- Using extended precision, we are able to get really tight enclosures for the fixed points and the associated constants.

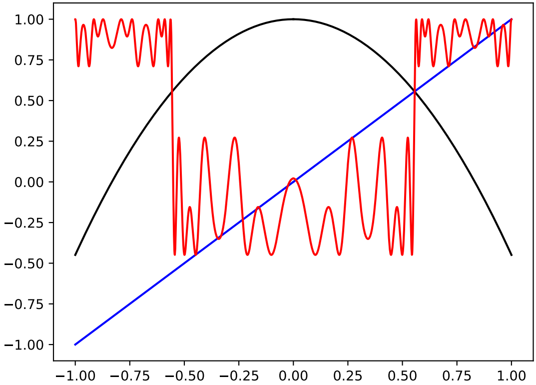
The classical $m = 2$ and $d = 2$ case

$$
\begin{aligned}
\delta = {} & 4.669201609102990671853203820466201617258185577475768632745651343 \\
& 004134330211314737138689744023948013817165984855189815134408627142027 \\
& 932522312442988890890859944935463236713411532481714219947455644365823 \\
& 793202009561058330575458617652222070385410646749494284981453391726200 \\
& 568755665952339875603825637225648004095107128389061184470277585428541 \\
& 980111344017500242858538249833571552205223608725029167886036267452721 \\
& 339905713160687534508343393444610370630945201911587697243227358983890 \\
& 3794946257251289097948 9 \pm 10^{-500},
\end{aligned}
$$

$$
\begin{aligned}
\alpha = {} & -0.3995352805231344898575804686336937194335442804669527275170730 \\
& 449124380166088380429818445948741812667617940648468383667140945404846 \\
& 164364373609475570184545976789402326870225485797735028209746477510392 \\
& 557978775073697474932326975513734923082122088541722241308330948027391 \\
& 890574703944646041606699384157782298900077729901354421213971924552385 \\
& 259444903372376975537750905488329754433672693681140505788840461793440 \\
& 186571478080760841608146499827233996549139348743626575822619683926231 \\
& 33476566990066713874212357 9 \pm 10^{-500}.
\end{aligned}
$$

- The proof uses $K = 680$ Chebyshev modes and 4096 digits of precision.

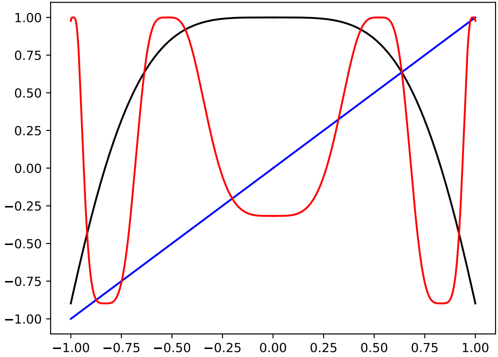
An example for $m = 10$ and $d = 2$



Theorem [B., Gonzalez and Mireles James 2026]

There exists an analytic even fixed point of R_{10}, behaving quadratically near 0, which is at most 10^{-18} away (in $\mathcal{C}^0$ norm) from this approximation.

$$\delta = 1110.537874176532781602 \pm 10^{-18}, \quad \alpha = 0.020848923604938857 \pm 10^{-18}.$$



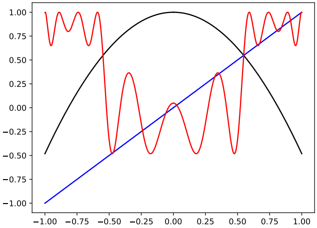
Theorem [B., Gonzalez and Mireles James 2026]

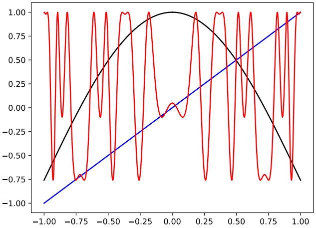
There exists an analytic even fixed point of R_3, behaving quartically near 0, which is at most 10^{-46} away (in $\mathcal{C}^0$ norm) from this approximation.

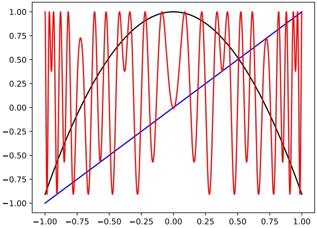
$$\delta = 85.7916290913560487440851221841293886768048969103 \pm 10^{-46},$$

$$\alpha = -0.3172430469535930185874716238506883387118289297 \pm 10^{-46}.$$

Some interesting observation (here for $m = 6$)







- Some of these fixed point seem to have the same universal constants.

Perspectives

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- There is a so-called PACSE theory,[8] which proves that if there are heteroclinic connections between two fixed points $f_m^\star$ and $f_n^\star$ of R_m and R_n, with associated universal constants δ_m and δ_n, then there exist an infinite sequence of renormalization fixed points $f_{k_j}^\star$ of R_{k_j}, whose universal constants δ_{k_j} can be approximated in terms of δ_m and δ_n.

[8] de la Llave, Olvera and Petrov 2011

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- The assumptions underlying these theory, namely the existence of heteroclinic connections, have never been verified.
- We plan to obtain such connections using computer-assisted proofs and the flexible set-up based on Chebyshev series that we used for studying the fixed points.

[8] de la Llave, Olvera and Petrov 2011

THANK YOU FOR YOUR ATTENTION!